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EMBEDDING THE HYPERSPACES OF CIRCLE-LIKE PLANE CONTINUA

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ABSTRACT. The author shows that the hyperspace of continua of a circle-like plane continuum is embeddable in E^3. The restriction to plane continua is necessary, for the hyperspace of a solenoid is homeomorphic to the cone over that solenoid and hence not embeddable in E^3.

In this paper we show that the hyperspace of continua of a circle-like, plane continuum can be embedded in E^3. A novel aspect of this proof is that much of the work is done in the differentiable category; in particular, we use Sard's Theorem to obtain a simple closed curve, the existence of which is not readily apparent in either the piecewise-linear or topological category. Our program consists of three parts:

(1) Reduce the problem to the problem of approximating a certain differentiable map G between disks by embeddings in E^3.

(2) Use Sard's Theorem to find a certain simple closed curve L in the domain of G so that we may approximate G in two stages—restricted as a map between annuli and restricted as a map between small disks.

(3) Since the disk is so small that it causes no trouble, approximate the map between annuli by a certain compact, monotone map, approximate the map between disks by a homeomorphism, paste the two maps back together, and complete the embedding using approximation theorems of Radó and Bing.

We also show that the restriction to plane continua is necessary, for the hyperspace of a solenoid is homeomorphic to the cone over that solenoid and hence not embeddable in E^3.

George Henderson [4] has recently shown that the hyperspace of continua of an arc-like (chainable) continuum is embeddable in E^3. This extended the result of W. R. R. Transue [11], who proved this only for the pseudo-arc, in answer to a question of A. C. Connor [2, p. 152].

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D denotes the unit disk, S the unit circle, and I the unit interval. We often identify S with the boundary of D. A circle-like continuum is the inverse limit of an inverse sequence of circles with surjective bonding maps. We let (S, g) denote such an inverse sequence. $C(X)$, the hyperspace of the continuum X, is studied in [5]. A map $g: S \to S$ induces a map $G: C(S) \to C(S)$ defined by $G(T) = g(T)$.

THEOREM 1. *The hyperspace of continua of a circle-like, plane continuum can be embedded in E^3.*

PROOF. Let $Y = \lim(S, g)$ be a circle-like, plane continuum. There is no loss in assuming that each bonding map is a differentiable map of degree 1. Identify $C(S)$ with D by the homeomorphism which sends a positively oriented arc $[\phi_1, \phi_2]$ in S to the point $(r, \phi) = (1 - (\phi_2 - \phi_1)/2\pi, \phi_1)$ in D. Jack Segal [10] has shown that $C(Y)$ is homeomorphic to $\lim(C(S), G)$, where G is the map induced by the bonding map g.

We prove that $C(Y)$ is embeddable in E^3 by showing that each bonding map G may be approximated by embeddings in E^3 [6]. Let $\epsilon > 0$. We express points in D by radial coordinates (r, ϕ). On the boundary of D, $G = g$, so we may use Lemma 3.11 of [7] to find a differentiable map $G_1: D \to D$ such that

(1) G_1 restricted to S is g,

(2) G_1 takes the origin to the origin, and

(3) G_1 is an $\epsilon/4$ approximation to G.

Furthermore, since G is an increasing function with respect to r, we see that by a judicious choice of the differentiable partition of unity in the proof of Lemma 3.11, we may assume that $\partial G_1/\partial r \geqq 0$.

Let δ be a positive number so small that G_1^{-1} of the disk centered at the origin of radius δ is contained in the disk centered at the origin of radius $\epsilon/16$. Let $\pi: D \to R^1$ be the projection map defined by $\pi(r, \phi) = r$. Then πG_1 is differentiable; hence by Sard's Theorem [8, p. 10], there exists a positive number $s < \delta$ such that s is a regular value of πG_1. Therefore, by Lemma 4 of [8], $(\pi G_1)^{-1}(s)$ is a smooth 1-manifold without boundary. Since $\partial G_1/\partial r \geqq 0$, $(\pi G_1)^{-1}(s)$ is a simple closed curve L.

Let $[0, L]$ denote the disk bounded by L, and let $[L, S]$ denote the annulus bounded by L and S. Consider $G_1 | [L, S]$ as a map $f: S^1 \times I \to S^1 \times I$ of an annulus onto itself.

We complete the proof by modifying and combining techniques of McCord [6] and Henderson [4]. Let $p: R^1 \times I \to S^1 \times I$ be the universal covering map $p(\phi, t) = (\exp 2\pi i\phi, t)$. Lift f to a map $\tilde{f}: R^1 \times I$

$\to R^1 \times I$ such that $p\bar{f} = fp$. Let $\pi_1: R^1 \times I \to R^1$ be the projection map. Note that $\{ |\pi_1 \bar{f}(\phi, t) - \phi| : (\phi, t) \in R^1 \times I \}$ is bounded by some integer N. Define $G_2: S \times I \to S \times I \times R^1$ by

$$G_2(p(\phi, t)) = (f(p(t)), (\pi_1 \bar{f}(\phi, t) - \phi)\epsilon/16N).$$

The map G_2 is well defined and continuous as well as an $\epsilon/16$-approximation to the map $\bar{f}: S^1 \times I \to S^1 \times I \times R^1$ defined by $\bar{f}(\theta, t) = (f(\theta, t), 0)$. Although G_2 is not 1-1 as we would like it to be, a simple calculation reveals that if $G_2(p(\phi_1, t_1)) = G_2(p(\phi_2, t_2))$, then $\phi_1 - \phi_2$ is an integer. Hence if $G_2(\theta_1, t_1) = G_2(\theta_2, t_2)$, then $\theta_1 = \theta_2$. Since $\partial G_1/\partial r \geqq 0$, G_2 maps the straight line segment between (θ_1, t_1) and (θ_2, t_2) to their common value. Therefore, point inverses of G_2 are points and straight line intervals.

Let K denote the circle centered at the origin of radius s, and M the disk bounded by K. We may regard G_2 as a map of $[L, S]$ into E^3. $G_2(L)$, a simple closed curve in the truncated cylinder $K \times (-\epsilon/16, \epsilon/16)$, bounds a disk contained in $M \times (-\epsilon/16, \epsilon/16)$. So extend G_2 to D by defining G_2 of $[0, L]$ to be this disk.

G_2 is now a map of a disk into E^3 such that point inverses are either points or straight line segments. Hence $G_2(D)$ is a disk. We use a theorem of Radó [**9**, Theorem 2.17] to find a homeomorphism G_3 of D onto $G_2(D)$ such that G_3 is an $\epsilon/4$-approximation to G_2. Finally, by [**3**], there is a homeomorphism G_4 of D into E^3 such that $G_4(D)$ is a polyhedron and G_4 is an $\epsilon/4$-approximation to G_3. So G_4 can be extended to a homeomorphism of E^3 onto itself. G_4 is the desired embedding which ϵ-approximates G. $\square$

The planar embedding of the circle-like continuum is essential in Theorem 1, as shown by the next theorem.

THEOREM 2. *The hyperspace of continua of a solenoid is homeomorphic to the cone over that solenoid and is therefore not embeddable in* E^3.

PROOF. We consider only the dyadic solenoid, defined as $\Sigma = \lim(S, w)$, where $w = z^2$. Consider the maps f, g of D onto D defined by

$$f(r, \theta) = (\max(2r - 1, 0), \theta),$$
$$g(r, \theta) = (r, 2\theta).$$

We see that $C(\Sigma)$ is homeomorphic to $\lim(D, fg)$ and that the cone over Σ is homeomorphic to $\lim(D, g)$. The following diagram gives a homeomorphism between $\lim(D, fg)$ and $\lim(D, g)$:

$$\begin{tikzcd}
D \arrow[d, "1"'] & D \arrow[l, "fg"'] \arrow[d, "1"'] & D \arrow[l, "fg"'] \arrow[d, "1"'] & \cdots \arrow[l] \\
D \arrow[d, "1"'] & D \arrow[l, "f_1 g"'] \arrow[d, "f_1"'] & D \arrow[l, "f_2 g"'] \arrow[d, "f_1 f_2"'] & \cdots \arrow[l, "f_3 g"'] \\
D & D \arrow[l, "g"'] & D \arrow[l, "g"'] & \cdots \arrow[l]
\end{tikzcd}$$

If we replace the collapse f by an appropriate homeomorphism (which depends only on r) at each stage and if these homeomorphisms approximate f sufficiently closely, then $\lim(D, fg)$ is homeomorphic to $\lim(D, f_n g)$. But since each f_n commutes with g, the sequence $1, f_1, f_1 f_2, \cdots, f_1 \cdots f_n, \cdots$ induces a homeomorphism between $\lim(D, f_n g)$ and $\lim(D, g)$. Finally Bennett and Transue [1] have shown that the cone over Σ, and hence $C(\Sigma)$, is not embeddable in E^3. $\square$

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