REAL-ANALYTIC SUBMANIFOLDS OF COMPLEX MANIFOLDS

L. R. HUNT

Abstract. This paper examines the extendibility of holomorphic functions on a real manifold which is embedded in a complex manifold. The principal result is that all real \( k \)-dimensional, real-analytic, compact manifolds embedded in an \( n \)-dimensional complex Stein manifold, where \( k > n \), are extendible over a manifold of one higher real dimension. A discussion is also given of the local equations of a manifold which is C-R in a neighborhood of some point.

1. Introduction. Wells [3] proves that there is an open dense set of embeddings of a real \( k \)-dimensional \( C^\infty \) compact manifold into an \( n \)-dimensional complex manifold \( X \), \( k > n \), with the property that each submanifold of \( X \) embedded by an element in this dense set is extendible over a real \( (k+1) \)-dimensional \( C^m \) manifold, \( 1 \leq m < \infty \). It is the purpose of this paper to show that all real \( k \)-dimensional, real-analytic, compact manifolds embedded in an \( n \)-dimensional complex Stein manifold \( X \), \( k > n \), are extendible over a manifold of one higher real dimension.

In §2 of this paper we give definitions of exceptional points, extendibility, and the Levi form. §3 contains a discussion of the local equations of a manifold which is C-R in a neighborhood of some point. A short discussion of peak points is given in §4, and we prove our result for real-analytic submanifolds of complex manifolds in §5.

2. Definitions. Let \( M^k \) be a real \( k \)-dimensional \( C^\infty \) manifold embedded in an \( n \)-dimensional complex manifold \( X \), where \( k, n \geq 2 \). If \( f \) is the embedding, then the real Jacobian of \( f \) has maximal rank. Let us denote the complex Jacobian of \( f \) by \( J(f) \) and set \( q = \min(n, k) \).

A point \( p \) in \( M^k \) is said to be an exceptional point of order \( l \), \( 0 \leq l \leq \left\lceil k/2 \right\rceil - \max(k - n, 0) \), if the complex rank of \( J(f) \big|_p \) is equal to \( q - l \).

A point \( p \) in \( M^k \) is generic if \( p \) is an exceptional point of order 0. The manifold \( M^k \) is locally generic at \( p \) in \( M^k \) if every point in some open neighborhood of \( p \) is generic, and is locally C-R at \( p \) if every point in some open neighborhood of \( p \) is an exceptional point of the

Received by the editors August 13, 1970.

AMS 1969 subject classifications. Primary 3220, 3225; Secondary 3227.

Key words and phrases. Extendibility of holomorphic functions, real-analytic submanifolds of complex manifolds, exceptional points, locally C-R.

Copyright © 1971, American Mathematical Society

69
same order. If \( k \leq n \), an exceptional point of order 0 is a **totally real** point.

Suppose \( M^k \) is locally C-R at \( p \), and \( p \) is not a totally real point for \( M^k \). Let \( T_x(M^k) \) be the real tangent space to \( M^k \) at \( x \), and \( H_x(M^k) \) be the space of holomorphic tangent vectors to \( M^k \) at \( x \). Then we define the Levi form, at any \( x \) near \( p \),

\[
L_x(M^k): H_x(M^k) \to (T_x(M^k) \otimes \mathbb{C})/(H_x(M^k) \otimes \mathbb{C})
\]

by \( L_x(M^k)(t) = \pi_x\{[Y, \overline{Y}]_x\} \), where \( Y \) is a local section of \( H(M^k) \) such that \( Y_x = t \), \([Y, \overline{Y}]_x\) is the Lie bracket evaluated at \( x \), and

\[
\pi_x: T_x(M^k) \otimes \mathbb{C} \to (T_x(M^k) \otimes \mathbb{C})/(H_x(M^k) \otimes \mathbb{C})
\]

is the projection.

Denote by \( \mathfrak{h}_x = \mathfrak{h} \) the sheaf of germs of holomorphic functions on the second countable complex manifold \( X \). Let \( K \) be a compact subset of \( X \) and \( U \) an open subset of \( X \) containing \( K \). We set \( \mathcal{O}(K) = \text{ind lim}_{U \supset K} \mathcal{O}(U) \) where \( \mathcal{O}(U) \) is the Fréchet algebra of holomorphic functions on \( U \). We say that \( K \) is extendible to a connected set \( K' \supset K \) if the map \( r: \mathfrak{h}(K') \to \mathfrak{h}(K) \) is onto.

3. **Local equations.** Let \( J^l(M^k, X) \) be the \( r \)-jet bundle with fiber \( J^l(k, n) \) (e.g. see [2]). We can identify \( J^l(k, n) \) with the set of all complex \((n \times k)\) matrices. Denote by \( S_l(k, n) = S_l \) the subset of \( J^l(k, n) \) with rank equal to \((q - l), 0 \leq l \leq [k/2] - \max(k - n, 0) \). Then if \( M^k \) is embedded in \( X \) by the map \( f \), a point \( p \) in \( M^k \) is an exceptional point of order \( l \) if and only if \( J(f) \big|_{\mathcal{O}(K)} \) is in \( S_l \).

Suppose the point \( p \) in \( M^k \) is an exceptional point of order \( l \). If \( k \geq n \), then the local equations of \( M^k \) in a neighborhood of \( p \) are given (after a proper coordinate change) by

\[
\begin{align*}
0_1 &= x_1 + ih_1(x_1, \ldots, x_{2(n-l) - k}, w_1, \ldots, w_{k-n+l}), \\
0_2 &= x_2(n-l-k) + ih_2(n-l-k)(x_1, \ldots, x_{2(n-l) - k}, w_1, \ldots, w_{k-n+l}), \\
0_{n-l+1} &= x_{n-l+1} + i\bar{v}_1 = \bar{w}_1, \\
0_{n-l+2} &= x_{n-l+2} + i\bar{v}_2 = \bar{w}_2, \\
\vdots
\end{align*}
\]

(1)

\[
\begin{align*}
0_{n-l} &= x_{n-l} + i\bar{w}_{k-n+l} = x_{n-l+k}, \\
0_{n-l+1} &= x_{n-l+1} + i\bar{w}_{k-n+l} = \bar{w}_{k-n+l}, \\
0_{n-l+2} &= x_{n-l+2} + i\bar{w}_{k-n+l} = \bar{w}_{k-n+l}, \\
\vdots
\end{align*}
\]

(1)
where \( x_1, \ldots, x_{2(n-l)-k}, u_1, v_1, \ldots, u_{k-n+l}, v_{k-n+l} \) are local coordinates for \( M^k \) in a neighborhood of \( p \) vanishing at \( p \), and \( z_1, \ldots, z_n \) are local coordinates for \( X \) in a neighborhood of \( p \) vanishing at \( p \). The real-valued functions \( h_1, \ldots, h_{2(n-l)-k} \) as well as the complex-valued functions \( g_1, \ldots, g_l \) vanish to order 2 at \( p \).

In [2] we state that \( S_l \) is a regular submanifold of \( J^1(k, n) \) of complex codimension \( (k-n+l)l \). However, we notice that \( (k-n+l)l \) is just the number of complex lines in the local equations times the number of functions \( g_i \) in the local equations.

The question we are interested in is the following. What form must the equations take in order for the embedded manifold to be locally C-R at the exceptional point \( p \) of order \( l \)? Since every point \( p' \) in some open neighborhood of \( p \) would have the property that \( J(f) \big|_{p'} \) must satisfy \( (k-n+l)l \) complex conditions, we find it is difficult (in the sense of transversality theory) for a manifold to be locally C-R except when \( l=0 \). Using equations (1) we find that our complex Jacobian at a point \( p' \) can be put in the form

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix},
\]

where \( A \) is the \((2(n-l)-k) \times (2(n-l)-k)\) identity matrix, \( B \) and \( C \) are both zero matrices, and \( D \) is the \((2l+k-n) \times (2(k-n+l))\) matrix

\[
\begin{bmatrix}
1 & i & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & i & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & i \\
\frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial v_1} & \frac{\partial g_1}{\partial u_2} & \frac{\partial g_1}{\partial v_2} & \cdots & \frac{\partial g_1}{\partial u_{k-n+l}} & \frac{\partial g_1}{\partial v_{k-n+l}} \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
\frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial v_1} & \frac{\partial g_1}{\partial u_2} & \frac{\partial g_1}{\partial v_2} & \cdots & \frac{\partial g_1}{\partial u_{k-n+l}} & \frac{\partial g_1}{\partial v_{k-n+l}}
\end{bmatrix}.
\]

Hence \( M^k \) is locally C-R at the exceptional point \( p \) of order \( l \) if and only if \( (\partial g_i/\partial u_j) = -i(\partial g_i/\partial v_j) \) holds in some neighborhood of \( p \) for \( j=1, \ldots, k-n+l \) and \( i=1, \ldots, l \). Thus for the manifold \( M^k \) to be locally C-R the functions \( g_1, \ldots, g_l \) in our local equations (1) must be complex-analytic functions of \( w_1, \ldots, w_{k-n+l} \).
Again let \( p \) be an exceptional point of order \( l \) in \( M^k \), but assume \( k < n \). The local equations of \( M^k \) in a neighborhood of \( p \) are given (after a proper coordinate change) by

\[
\begin{align*}
Z_1 &= x_1 + ih_1(x_1, \ldots, x_{k-2l}, w_1, \ldots, w_l), \\
& \quad \vdots \\
Z_{k-2l} &= x_{k-2l} + ih_{k-2l}(x_1, \ldots, x_{k-2l}, w_1, \ldots, w_l), \\
Z_{k-2l+1} &= u_1 + iv_1 = w_1, \\
& \quad \vdots \\
Z_{k-l} &= u_l + iv_l = w_l, \\
Z_{k-l+1} &= g_1(x_1, \ldots, x_{k-2l}, w_1, \ldots, w_l), \\
& \quad \vdots \\
Z_n &= g_{n-k+l}(x_1, \ldots, x_{k-2l}, w_1, \ldots, w_l),
\end{align*}
\]

where \( x_1, \ldots, x_{k-2l}, u_1, v_1, \ldots, u_l, v_l \) are the local coordinates for \( M^k \) in a neighborhood of \( p \) vanishing at \( p \), and \( Z_1, \ldots, Z_n \) are the local coordinates for \( X \) in a neighborhood of \( p \) vanishing at \( p \).

We have that \( S_1 \) is a regular submanifold of \( J^l(k, n) \) of complex codimension \( (n-k+l) l \). In order for the manifold to be locally C-R at \( p \), a discussion similar to the previous one tells us that the functions \( g_1, \ldots, g_{n-k+l} \) in our local equations (2) must be complex-analytic functions of \( w_1, \ldots, w_l \).

4. Peak points and the Levi form. Let \( K \) be a compact subset of a complex manifold \( X \). We call a point \( x \in K \) a holomorphic peak point if there exists a function \( f \in \mathcal{O}(K) \) such that, for any \( y \in K - \{x\} \), we have \( |f(y)| < |f(x)| \).

In [3] it is shown that a compact \( C^2 \) submanifold of a Stein manifold \( X \) contains a nonempty open set of peak points. Also the Levi form cannot vanish at a peak point without contradicting the maximum principle.

5. Real-analytic manifolds. In this section we shall prove the asserted result for real-analytic manifolds.

**Theorem 5.1.** Let \( M^k \) be a real \( k \)-dimensional, real-analytic, compact manifold embedded in an \( n \)-dimensional Stein manifold \( X \), \( k > n \). Then \( M^k \) is extendible to a \( C^m \) submanifold \( \tilde{M} \) of \( X \) with real dimension \( \tilde{M} = k + 1 \) and \( 1 \leq m < \infty \).

**Proof.** Given any point \( p \) in \( M^k \), arbitrarily close to \( p \) we can find an open set of points at which \( M^k \) is locally C-R. This follows from
the fact that if $M^k$ is not locally C-R at $p$, then arbitrarily close to $p$ we can find a point which is an exceptional point of order less than the order of $p$. We repeat this process until we find a point at which $M^k$ is locally C-R (for a “dense” set of embeddings this will mean locally generic).

Let $p$ be a peak point in the open set mentioned in §4. Then there exists a point $p'$ in $M^k$ and an open neighborhood of this point on which $M^k$ is locally C-R and the Levi form does not vanish. If $M^k$ is locally generic at $p'$, we have the conditions for local extensibility given in [1] and [4] and the proof is complete. If $M^k$ is locally C-R at $p'$ and not generic there we have the local equations (1) where $l > 0$ and $g_1, \ldots, g_l$ are complex-analytic functions of $w_1, \ldots, w_{k-n+l}$.

Since our manifold is real-analytic, we expand each function $g_i$ in a power series

$$g_i(x, w) = \sum_a a_{i,a}(x, w)^a,$$

where $x = (x_1, \ldots, x_2(n-l)-k, w_1, \ldots, w_{k-n+l})$, and $\alpha = (\alpha_1, \ldots, \alpha_{n-l})$. Replacing $z_j$ by $z_j = \sum_a a_{i,a} \sigma^a$, where $z = (z_1, \ldots, z_{n-l})$, $j = n-l+1, \ldots, n$, and $j = n-l+i$, we find in our new coordinates that each $a_{i,a} = 0$. Hence our local equations in our new coordinate system are

$$z_1 = x_1 + ih_1(x_1, \ldots, x_2(n-l)-k, w_1, \ldots, w_{k-n+l}),$$
$$z_2(n-l)-k = x_2(n-l)-k + ih_2(n-l)-k(x_1, \ldots, x_2(n-l)-k, w_1, \ldots, w_{k-n+l}),$$
$$z_2(n-l)-k+1 = u_1 + iv_1 = w_1,$$
$$\vdots$$
$$z_n = u_{k-n+l} + iv_{k-n+l} = w_{k-n+l},$$
$$z_{n-l+1} = 0,$$
$$\vdots$$
$$z_n = 0.$$

Thus our point $p'$ appears locally as a generic point of a real $k$-dimensional submanifold of $C^n$. Since the Levi form’s vanishing or nonvanishing is independent of coordinate changes, we have a generic point at which the Levi form does not vanish. This establishes the extendibility as given in the statement of the theorem.

Remark. The theorem is false if $k \leq n$ because of the existence of totally real submanifolds which are always holomorphically convex.
REFERENCES


Texas Tech University, Lubbock, Texas 79409