A CLASS OF EMBEDDINGS OF $S^{n-1}$ AND $B^n$ IN $R^n$

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Abstract. We show that if $D$ is an $n$ or $(n-1)$-cell in $R^n$, $n \geq 4$, and $E$ is an $(n-2)$-cell in Bd $D$, with $D-E$ locally flat in $R^n$ and $E$ locally flat in each of Bd $D$ and $R^n$, then $D$ is locally flat in $R^n$.

In establishing criteria for detecting local flatness of submanifolds, a central role has been played by the following problem [1], [2].

$\gamma(n, m, k)$: If $D$ is an $m$-cell in $R^n$, $E$ is a $k$-cell in Bd $D$, and if $D-E$ is locally flat in $R^n$ and $E$ is locally flat in both $R^n$ and Bd $D$, then $D$ is locally flat in $R^n$.

$\gamma(n, n, n-2)$ and $\gamma(n, n-1, n-2)$, $n \geq 3$, are the only unresolved $\gamma$-statements, and, for fixed $n$, these are known to be equivalent [1]. In this paper we show that $\gamma(n, n, n-2)$ is true for $n \geq 4$. Of equal importance is the illustration of the utility of the 1-SS property introduced in [3].

Definition. Let $X \subseteq Y$ be topological spaces. Then $Y-X$ is said to be 1-SS (1-short shrink) at $x \in X$ if for every neighborhood $U$ of $x$ there is a neighborhood $V \subseteq U$ of $x$ such that every loop in $V-X$ which is null-homotopic in $Y-X$ is also null-homotopic in $U-X$.

Theorem 1. Suppose that $S^{n-1} \subseteq S^n$ is an $(n-1)$-sphere and that $D^{n-2} \subseteq S^{n-1}$ is an $(n-2)$-cell. If $S^{n-1}-D^{n-2}$ and $D^{n-2}$ are locally flat in $S^n$, and $D^{n-2}$ is locally flat in $S^{n-1}$ then $S^n-S^{n-1}$ is 1-LC at each point of $S^{n-1}$.

Before proving the theorem we will establish the following lemma.

Lemma 1. Let $S^{n-2} \subseteq S^n$ be an $(n-2)$-sphere which is locally flat at a point $x$. Then, $S^n-S^{n-2}$ is 1-SS at $x$.

Proof. Let $U$ be any neighborhood of $x$ and let $V$ be a subset of $U$ that is a flattening cell neighborhood for $S^{n-2}$ at $x$, i.e., $(V, V \setminus S^{n-2}) \approx (I^n, I^{n-2})$. Let $l$ be a loop in $V-S$ which is null-homotopic in $S^n-S$. By pushing radially away from $x$ we see that $l$ is homotopic in $V-S$ to a loop $l'$ in Bd $V-S$ which is null-homotopic in $S^n-(\text{Int } V\cup S)$. The proof will be complete if we can show that $l'$ is null-homotopic in $S^n-S^{n-2}$.
Bd $V - \Sigma$. Since we know that $l'$ is null-homotopic in $S^n - (\text{Int } V \cup \Sigma)$, it will suffice to show that the injection

$$\pi_1(\text{Bd } V - \Sigma) \to \pi_1(S^n - (\text{Int } V \cup \Sigma))$$

is a monomorphism. In order to do this consider the following portion of the Mayer-Vietoris sequence

$$\cdots \to H_2(S^n - (\Sigma - \text{Int } V)) \to H_1(\text{Bd } V - \Sigma) \to H_1(S^n - (\text{Int } V \cup \Sigma))$$

$$\oplus H_1(V - (\text{Bd } V \cap \Sigma)) \to H_1(S^n - (\Sigma - \text{Int } V)) \to \cdots.$$ 

By using Alexander duality this sequence becomes

$$\cdots \to 0 \to Z \to Z \oplus 0 \to 0 \to \cdots.$$ 

Hence, the inclusion of Bd $V - \Sigma$ into $S^n - (\text{Int } V \cup \Sigma)$ induces an isomorphism on first homology. But now any loop $l$ in Bd $V - \Sigma$ which is null-homotopic in $S^n - (\text{Int } V \cup \Sigma)$ is also null-homologous in $S^n - (\text{Int } V \cup \Sigma)$, consequently null-homologous in Bd $V - \Sigma$. Since $\pi_1(\text{Bd } V - \Sigma)$ is abelian, it follows that $l$ is null-homotopic in Bd $V - \Sigma$ and so the injection $\pi_1(\text{Bd } V - \Sigma) \to \pi_1(S^n - (\text{Int } V \cup \Sigma))$ is a monomorphism as desired.

**Proof of Theorem 1.** Clearly $S^n - \Sigma^{n-1}$ is 1-LC at each point of $\Sigma^{n-1} - D^{n-2}$, since $\Sigma^{n-1}$ is locally flat at such points. Now suppose $x \in \text{Bd } D^{n-2}$. Let $U$ be any neighborhood of $x$ in $S^n$ and let $V \subseteq U$ be a flattening neighborhood for $D^{n-2}$ in $S^n$ at $x$, i.e., $(V, V \cap D^{n-2}) \approx (E^n, E^n_{n-2})$. Without loss of generality, we may assume that $V \cap \Sigma^{n-1} \subseteq B^{n-1}$ where $B^{n-1} \subseteq U$ is a flattening open $(n-1)$-cell neighborhood of $D^{n-2}$ in $\Sigma^{n-1}$ at $x$. Let $l : \text{Bd } I^2 \to V - \Sigma^{n-1}$ be any loop, and let $f : I^2 \to V - D^{n-2}$ be an extension of $l$. Clearly, there is a closed $(n-1)$-cell $D_0 \subseteq B^{n-1} - D^{n-2}$ such that $f(I^2) \cap \Sigma^{n-1} \subseteq D_0$. Let $G$ denote the closure of the complementary domain of $S^{n-2}$ in $S^n$ which does not contain $l(\text{Bd } I^2)$. Let $A = f^{-1}(G)$. Then, by Tietze’s extension theorem $f \mid A \cap f^{-1}(\Sigma^{n-1})$ can be extended to a map $f' : A \to D_0$. Redefine $f$ to be $f'$ on $A$. By using a collar of $D_0$ in $\text{Cl}(S^n - G)$ (which exists since $D_0$ is locally flat), we can “pull in” $f$ to obtain $f_* : I^2 \to U - \Sigma^{n-1}$ and so $l$ is null-homotopic in $U - \Sigma^{n-1}$. Hence, $S^n - \Sigma^{n-1}$ is 1-SC at $x$.

Suppose that $x \in \text{Int } D^{n-2}$. Since $D^{n-2}$ is locally flat in $\Sigma^{n-1}$, we may complete $D^{n-2}$ to an $(n-2)$-sphere $\Sigma^{n-2} \subseteq \Sigma^{n-1}$. Let $U$ be any neighborhood of $x$ in $S^n$, let $B^{n-1} \subseteq U$ be a flattening $(n-1)$-cell neighborhood of $\Sigma^{n-2}$ in $\Sigma^{n-1}$ at $x$, and let $U' \subseteq U$ be a neighborhood of $x$ in $S^n$ such that $U' \cap \Sigma^{n-1} \subseteq B^{n-1}$. Since $S^n - \Sigma^{n-2}$ is 1-SS at $x$ by the lemma, there is a neighborhood $V \subseteq U'$ of $x$ such that every loop in $V - \Sigma^{n-2}$ which is null-homotopic in $S^n - \Sigma^{n-2}$ is also null-homotopic in $U' - \Sigma^{n-2}$. Let $l : \text{Bd } I^2 \to V - \Sigma^{n-1}$ be any loop. Since $D^{n-2}$ is flat in
there is a map \( f : I^2 \to S^n - D^{n-2} \) which extends \( l \). Obviously, there is a closed, locally flat \((n-1)\)-cell \( D_0 \subseteq \Sigma^{n-1} - D^{n-2} \) such that \( f(I^2) \cap \Sigma^{n-1} \subseteq D_0 \). By making an application of Tietze’s extension theorem similar to the one in the preceding paragraph, we can obtain \( f_* : I^2 \to S^n - \Sigma^{n-1} \) which extends \( l \). But this means that \( l \) is null-homotopic in \( S^n - \Sigma^{n-2} \) and so by our choice of \( V \), \( l \) is null-homotopic in \( U' - \Sigma^{n-2} \), i.e., there is a map \( g : I^2 \to U - \Sigma^{n-2} \) which extends \( l \). Clearly, there are two closed, locally flat \((n-1)\)-cells \( D_+ \) and \( D_- \) in \( B^{n-1} - \Sigma^{n-2} \) such that \( g(I^2) \cap \Sigma^{n-1} \subseteq D_+ \cup D_- \). Let \( G \) denote the complementary domain of \( \Sigma^{n-1} \) in \( S^n \) which contains \( \partial (Bd I^2) \). Let \( X \) denote the component of \( g^{-1}(G) \) which contains \( \partial (Bd I^2) \) and consider the components of \( I^2 - X \). Let \( A_+ \) be the union of all those components having frontiers whose images are contained in \( D_+ \) and let \( A_- \) be the union of all those components having frontiers whose images are contained in \( D_- \). (By unicoherence these frontiers are connected and so their images are contained in either \( D_+ \) or \( D_- \).) Then, by Tietze’s extension theorem \( g|_{A_+} \cap g^{-1}(\Sigma^{n-1}) \) can be extended to a map \( g_+: A_+ \to D_+ \) and \( g|_{A_-} \cap g^{-1}(\Sigma^{n-1}) \) can be extended to a map \( g_- : A_- \to D_- \). Redefine \( g \) to be \( g_+ \) on \( A_+ \) and \( g_- \) on \( A_- \). By using a collar of \( D_+ \) and \( D_- \) in \( \text{Cl}(G \cap U) \) (which exist since \( D_+ \) and \( D_- \) are locally flat), we can “pull in” \( g \) to obtain \( g_* : I^2 \to U - \Sigma^{n-1} \) and so \( l \) is null-homotopic in \( U - \Sigma^{n-1} \). Hence, \( \Sigma^{n-1} \) is 1-LC at \( x \) as desired.

**Theorem 2.** \( \gamma(n, n, n-2) \) is true.

**Proof.** Let \( D^n \) and \( E^{n-2} \), \( n > 4 \), be as in the statement of \( \gamma(n, n, n-2) \). It suffices to show that \( \partial D \) is locally flat. By [4], this will be the case if \( \partial D \) can be pointwise approximated by locally flat spheres and \( R^n - \partial D \) is 1-LC at each point of \( \partial D \). The first condition follows from the fact that \( \partial D \) is collared on one side, and the second follows from Theorem 1.

**Bibliography**


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