SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

SERIES CONVERGENCE ON BOOLEAN ALGEBRAS

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Abstract. Necessary and sufficient conditions for convergence of an infinite series \( \sum_{n=1}^{\infty} a_n \) of elements of an abstract boolean \( \sigma \)-algebra is that \( \lim_{n \to \infty} a_n = 0 \).

We will base our investigation on the monograph [1] by Carathéodory, and define an abstract boolean \( \sigma \)-algebra \( B \) as an idempotent ring with a unit which is \( \sigma \)-complete (see also [3]). Let us denote addition by \( \sum \), multiplication by \( \cap \) and the corresponding join by \( \cup \).

Further, if \( x_n \in B, n = 1, 2, 3, \ldots \), then \( x^n = \limsup x_n = \bigcap_{i=1}^{\infty} \bigcup_{n=i}^{\infty} x_i, \ x = \lim_{n \to \infty} x_n \in B \) def \( x^n = x \).

Finally we define \( \sum_{n=1}^{\infty} x_n = \lim_{n \to \infty} \sum_{i=1}^{n} x_i. \)

Theorem. Let \( B \) be an abstract boolean \( \sigma \)-algebra, \( a_n \in B \) for \( n = 1, 2, 3, \ldots \). Then \( \sum_{n=1}^{\infty} a_n \) exists if and only if \( \lim_{n \to \infty} a_n = 0 \).

Proof. The following statements are consequences of properties of the operations \( \sum, \cup, \cap \) and can be found proven for example in §§1–32 of [1]. Necessity of the condition is an obvious consequence of continuity of \( \sum \). Let \( a'_n = 1 + a_n, \ d_n = \sum_{i=1}^{n} a_i. \) Then

\[
1 = \limsup a_n + \liminf a'_n
\]

\[
= 0 + \liminf a'_n = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} a'_i;
\]

\[
\limsup d_n = 1 \cap \limsup d_n = \left( \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} a'_i \right) \cap \left( \bigcap_{m=1}^{\infty} \bigcup_{j=m}^{\infty} d_j \right)
\]

\[
= \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{j=m}^{\infty} (a'_i \cap d_j) \subset \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} (a'_i \cap d_j).
\]
\[
\bigcap_{i=n}^{\infty} \bigcup_{j=n}^{\infty} (a_i \cap d_j) = \bigcup_{j=n}^{\infty} \bigcap_{i=n}^{\infty} a'_i \cap \left[ \left( \sum_{k=1}^{j-1} a_k \right) \cap \left( \bigcap_{i=n}^{j} a'_i \right) \right] \\
+ \left( \sum_{k=n}^{j} a_k \right) \cap \left( \bigcap_{i=n}^{j} a'_i \right) \\
= \bigcup_{j=n}^{\infty} \bigcap_{i=n}^{\infty} a'_i \cap \left[ d_{n-1} \cap \bigcap_{i=n}^{j} a'_i + \sum_{k=n}^{j} (a_k \cap a'_i) \right] \\
= d_{n-1} \cap \bigcup_{j=n}^{\infty} \bigcap_{i=n}^{\infty} a'_i = d_{n-1} \cap \bigcap_{i=n}^{\infty} a'_i
\]

because \( \sum_{k=n}^{j} (a_i \cap a_k) = 0 \).

For \( k \geq n \),
\[
d_k \cap \bigcap_{i=n}^{\infty} \bigcup_{j=n}^{\infty} (a'_i \cap d_j) = d_k \cap d_{n-1} \cap \bigcap_{i=n}^{\infty} a'_i \\
= d_{n-1} \cap \left( d_{n-1} + \sum_{i=n}^{k} a_i \right) \cap \bigcap_{i=n}^{\infty} a'_i \\
= d_{n-1} \cap \bigcap_{i=n}^{\infty} a'_i \cap \left( d_{n-1} \cap \bigcap_{i=n}^{k} a'_i + \left( \sum_{i=n}^{k} a_i \right) \cap \left( \bigcap_{i=n}^{k} a'_i \right) \right) \\
= d_{n-1} \cap \bigcap_{i=n}^{\infty} a'_i 
\]

Therefore \( \bigcap_{i=n}^{\infty} \bigcup_{j=n}^{\infty} (a'_i \cap d_j) \subseteq d_k \) for every \( k \geq n \). Finally we have
\[
\limsup d_n \subseteq \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} \bigcup_{j=n}^{\infty} (a'_i \cap d_j) \subseteq \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} d_k = \liminf d_n. \quad \text{Q.E.D.}
\]

COMMENT. Apparently existence of the unit in \( B \) is not substantial. All we need is the existence of an upper bound \( m \) for any sequence of elements \( a_n \) in \( B \).

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REFERENCES


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