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SHORTER NOTES

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SERIES CONVERGENCE ON BOOLEAN ALGEBRAS

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ABSTRACT. Necessary and sufficient conditions for convergence of an infinite series $\sum_{n=1}^{\infty} a_n$ of elements of an abstract boolean σ-algebra is that $\lim_{n\to\infty} a_n = 0$.

We will base our investigation on the monograph [1] by Carathéodory, and define an abstract boolean σ-algebra $\boldsymbol{B}$ as an idempotent ring with a unit which is σ-complete (see also [3]). Let us denote addition by $\sum$, multiplication by $\bigcap$ and the corresponding join by $\bigcup$.

Further, if $x_n \in \boldsymbol{B}$, $n = 1, 2, 3, \cdots$, then $x^{\sim} = \limsup x_n = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} x_i$, $x_{\sim} = \liminf x_n = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} x_i$, $x = \lim_{n\to\infty} x_n \Leftrightarrow_{\text{def}} x^{\sim} = x_{\sim} = x$.

Finally we define $\sum_{n=1}^{\infty} x_n = \lim_{n\to\infty} \sum_{i=1}^{n} x_i$.

THEOREM. *Let $\boldsymbol{B}$ be an abstract boolean σ-algebra, $a_n \in \boldsymbol{B}$ for $n = 1, 2, 3, \cdots$. Then $\sum_{n=1}^{\infty} a_n$ exists if and only if* $\lim_{n\to\infty} a_n = 0$.

PROOF. The following statements are consequences of properties of the operations $\sum$, $\bigcup$, $\bigcap$ and can be found proven for example in §§1–32 of [1]. Necessity of the condition is an obvious consequence of continuity of $\sum$. Let $a_n' = 1 + a_n$, $d_n = \sum_{i=1}^{n} a_i$. Then

$$\begin{aligned} 1 &= \limsup a_n + \liminf a_n' \\ &= 0 + \liminf a_n' = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} a_i' ; \end{aligned}$$

$$\begin{aligned} \limsup d_n &= 1 \cap \limsup d_n = \left(\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} a_i' \right) \cap \left(\bigcap_{m=1}^{\infty} \bigcup_{j=1}^{\infty} d_j \right) \\ &= \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{j=m}^{\infty} (a_i' \cap d_j) \subset \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} \bigcup_{j=n}^{\infty} (a_i' \cap d_j). \end{aligned}$$

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$$\bigcap_{i=n}^{\infty} \bigcup_{j=n}^{\infty} (a_i' \cap d_j) = \bigcup_{j=n}^{\infty} \bigcap_{i=j+1}^{\infty} a_i' \cap \left[\left(\sum_{k=1}^{n-1} a_k\right) \cap \left(\bigcap_{i=n}^{j} a_i'\right) + \left(\sum_{k=n}^{j} a_k\right) \cap \left(\bigcap_{i=n}^{j} a_i'\right)\right]$$

$$= \bigcup_{j=n}^{\infty} \bigcap_{i=j+1}^{\infty} a_i' \cap \left[d_{n-1} \cap \bigcap_{i=n}^{j} a_i' + \sum_{k=n}^{j} \bigcap_{i=n}^{j} (a_k \cap a_i')\right]$$

$$= d_{n-1} \cap \bigcup_{j=n}^{\infty} \bigcap_{i=n}^{\infty} a_i' = d_{n-1} \cap \bigcap_{i=n}^{\infty} a_i'$$

because $\sum_{k=n}^{j} \bigcap_{i=n}^{j} (a_i' \cap a_k) = 0$.

For $k \geqq n$,

$$d_k \cap \bigcap_{i=n}^{\infty} \bigcup_{j=n}^{\infty} (a_i' \cap d_j) = d_k \cap d_{n-1} \cap \bigcap_{i=n}^{\infty} a_i'$$

$$= d_{n-1} \cap \left(d_{n-1} + \sum_{l=n}^{k} a_l\right) \cap \bigcap_{i=n}^{\infty} a_i'$$

$$= d_{n-1} \cap \bigcap_{i=k+1}^{\infty} a_i' \cap \left(d_{n-1} \cap \bigcap_{i=n}^{k} a_i' + \left(\sum_{l=n}^{k} a_l\right) \cap \left(\bigcap_{i=n}^{k} a_i'\right)\right)$$

$$= d_{n-1} \cap \bigcap_{i=n}^{\infty} a_i'.$$

Therefore $\bigcap_{i=n}^{\infty} \bigcup_{j=n}^{\infty} (a_i' \cap d_j) \subset d_k$ for every $k \geqq n$. Finally we have

$$\limsup d_n \subset \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} \bigcup_{j=n}^{\infty} (a_i' \cap d_j) \subset \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} d_k = \liminf d_n . \quad \text{Q.E.D.}$$

COMMENT. Apparently existence of the unit in $\boldsymbol{B}$ is not substantial. All we need is the existence of an upper bound m for any sequence of elements a_n in $\boldsymbol{B}$.

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