PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 29, Number 1, June 1971

# TRACE-CLASS FOR A HILBERT MODULE

GEORGE R. GIELLIS[1]

ABSTRACT. Let $H$ be a Hilbert module over a proper $H^*$-algebra $A$, and let $\tau(H)=\{fa:f\in H,\ a\in A\}$. Then we define a Banach space norm on $\tau(H)$ so that the module operation is continuous with respect to both variables. $\tau(H)$ is shown to be the dual of a certain space of bounded operators from $H$ to $A$, and the dual of $\tau(H)$ is also identified.

1. **Introduction.** For a proper $H^*$-algebra $A$, the trace-class of $A$ is defined as $\tau(A)=\{ab\mid a,\ b\in A\}$. The theory of the trace class was developed by Saworotnow and Friedell [6]. Generalizing the work of Schatten [3], they defined a norm on $\tau(A)$ so that $\tau(A)$ is a Banach algebra in this norm. Saworotnow [5] continued this development by showing that the dual of $\tau(A)$ is the space of all right centralizers on $A$, and that $\tau(A)$ is the dual of a certain space of right centralizers on $A$.

In this paper we consider a Hilbert module $H$ over $A$ and the set $\tau(H)=\{fa\mid f\in H,\ a\in A\}$. We introduce a Banach space norm on $\tau(H)$ for which the module operation is continuous in both variables. We also formulate and prove duality theorems analogous to those of Saworotnow.

2. **Definitions, notation and statement of results.** Throughout this paper $A$ will denote a proper $H^*$-algebra, with trace-class $\tau(A)=\{ab\mid a,b\in A\}$. A projection in $A$ is a nonzero selfadjoint idempotent element of $A$. By an orthogonal projection base (OPB) for $A$ is meant a maximal family of mutually orthogonal projections in $A$. If $\{e_\alpha\}$ is an OPB, then $A=\oplus\sum_\alpha Ae_\alpha=\oplus\sum_\alpha e_\alpha A$.

The trace functional on $\tau(A)$ is denoted as tr( ). We have $\operatorname{tr}(ab)=(a,b^*)=\sum_\alpha(abe_\alpha,e_\alpha)$ where $\{e_\alpha\}$ is an OPB for $A$. The trace norm $\tau(\ )$ on $\tau(A)$ is defined by $\tau(a)=\operatorname{tr}([a])$ where $[a]^2=a^*a$.

$H$ will denote a Hilbert module over $A$, i.e., $H$ is a right module over $A$ with a map $(\ ,\ )$ of $H\times H$ into $\tau(A)$, which is called the vector inner product for $H$. The properties of $(\ ,\ )$ generalize the properties of a Hilbert space inner product, e.g. additivity, $(f,g)^*=(g,f)$ and

Received by the editors August 19, 1970.

*AMS* 1970 *subject classifications*. Primary 46H25, 47B10; Secondary 46K15, 46C10.

*Key words and phrases*. $H^*$-algebra, orthogonal projection base, trace norm, centralizer, Hilbert module, vector inner product.

[1] This research was supported by N.S.F. Grant GP-11118.

$(f, ga) = (f, g)a$ for all $f, g \in H$ and $a \in A$. $H$ also has a linear structure and is a Hilbert space in the inner product $[f, g] = \mathrm{tr}(g, f)$ for $f, g \in H$. For a detailed discussion on Hilbert modules the reader is referred to [4].

We define the trace-class of $H$ as $\tau(H) = \{fa \mid f \in H, a \in A\}$. For a nonzero element $f$ of $H$, there exists a unique positive element $a$ of $A$ such that $a^2 = (f, f)$. We will denote this element as $a = [f]$. It will be shown that $f \in \tau(H)$ iff $[f] \in \tau(A)$.

THEOREM 1. *For $f \in \tau(H)$ define $\pi(f) = \tau([f])$. Then $\tau(H)$ is a linear space and $\pi(\ )$ is a Banach space norm on $\tau(H)$ such that $\pi(fa) \leq \|f\| \, |a|$ for all $f \in H$, $a \in A$.*

We shall make use of the following spaces of mappings:

$$R(A) = \{T: A \to A \mid T(ab) = (Ta)b \text{ for all } a, b \in A\}.$$
$$R(AH) = \{T: A \to H \mid T(ab) = (Ta)b \text{ for all } a, b \in A\}.$$
$$R(HA) = \{T: H \to A \mid T(fa) = (Tf)a \text{ for all } f \in H, a \in A\}.$$

Each of the above maps is necessarily linear and continuous. A proof of this in the case of $R(A)$ is given in [7]. We shall also assume such results as $T \in R(A)$, $S \in R(HA)$ implies $TS \in R(HA)$ and $T \in R(HA)$ iff $T^* \in R(AH)$.

For $a \in A$, let $L_a$ denote the operator $x \to ax$ $(x \in A)$ and let $C(A)$ be the closure in the operator norm of the space of all $L_a$, $a \in A$. Saworotnow's duality theorems show that $\tau(A)$ is the dual of $C(A)$ and $R(A)$ is the dual of $\tau(A)$. For $f \in H$, let $L_f$ denote the map $g \to (g, f)$. Then $L_f \in R(HA)$ for each $f \in H$. Let $C(HA)$ be the closure in the operator norm of the space of all $L_f$, $f \in H$.

THEOREM 2. *For $f \in \tau(H)$, define $p_f(T) = \mathrm{tr}(Tf)$ $(T \in C(HA))$. Then the map $f \to p_f$ is an isometric isomorphism of $\tau(H)$ onto the space of all bounded linear functionals on $C(HA)$.*

THEOREM 3. *For $S \in R(HA)$, define $p_S(f) = \mathrm{tr}(Sf)$ $(f \in \tau(H))$. Then the map $S \to p_S$ is an isometric isomorphism of $R(HA)$ onto the space of all bounded linear functionals on $\tau(H)$.*

3. **Proof of Theorem 1.** The following basic lemmas will allow us to use the techniques of [6] to prove Theorem 1.

LEMMA 1. *For each nonzero $f \in H$, there exists a partial isometry $W \in R(AH)$ such that $f = W[f]$ and $W^*f = [f]$.*

PROOF. From [6], we know there exists a sequence $\{e_k\}$ of mutually orthogonal projections in $A$ and a sequence $\{\lambda_k\}$ of positive real numbers such that $[f]=\sum_k \lambda_k e_k$. For $x\in A$, define $Wx=\sum_k \lambda_k^{-1} f e_k x$. Then

$$\begin{aligned}[\lambda_k^{-1} f e_k x, \lambda_j^{-1} f e_j x] &= \mathrm{tr}(\lambda_j^{-1} f e_j x, \lambda_k^{-1} f e_k x) \\ &= (\lambda_j\lambda_k)^{-1}\,\mathrm{tr}(x^* e_j (f,f) e_k x) = 0 \quad \text{if } j \neq k,\end{aligned}$$

since $(f,f)e_k=[f]^2 e_k=\lambda_k^2 e_k$ for each $k$. A similar calculation yields

$$\|Wx\|^2 = \sum_k \|\lambda_k^{-1} f e_k x\|^2 = \sum_k |e_k x|^2.$$

Thus $W$ is a partial isometry with initial domain the closed right ideal $M=\oplus\sum_k e_k A$. It is easily verified that $W[f]=f$.

REMARK. If we let $T_f$ denote the bounded linear operator $a\to fa$, we know that $T_f$ has a unique decomposition $T_f=UP$, where $U$ is a partial isometry from $A$ to $H$ and $P=(T_f^* T_f)^{1/2}$. To verify that $U=W$ and $P=L_{[f]}$, it is sufficient to check that the null space of $W$ equals the null space of $L_{[f]}$ (see p. 68 of [1]). This is clear, since $[f]x=0$ iff $e_k x=0$ for each $k$ iff $Wx=0$.

LEMMA 2. *The following are equivalent*:
(i) $f\in\tau(H)$.
(ii) $[f]\in\tau(A)$.
(iii) *There exists a positive element $b$ of $A$ such that* $b^2=[f]$.
(iv) $\sum_\alpha ([f]e_\alpha, e_\alpha)<\infty$, *where* $\{e_\alpha\}$ *is an OPB for* $A$.

PROOF. The equivalence of the last three statements is shown in [6]. If $f=ga$, then $[f]=W^*f=W^*(ga)=(W^*g)a\in\tau(A)$. If $[f]\in\tau(A)$ and $b^2=[f]$, then $f=W(b^2)=(Wb)b\in\tau(H)$.

LEMMA 3. *If $T\in R(A)$ and $a\in\tau(A)$ then $\tau(Ta)\leqq\|T\|\tau(a)$. (This is Lemma 5 of* [6].)

Using the above lemmas, one can proceed as in [6] to show that $\tau(H)$ is a linear space and that $\pi(\ )$ defines a norm on $\tau(H)$. As an example we will show that $\pi(f+g)\leqq\pi(f)+\pi(g)$. Let $W$, $W_1$ and $W_2$ be partial isometries as in Lemma 1 such that $W[f+g]=f+g$, $W_1[f]=f$ and $W_2[g]=g$. Then we have

$$\begin{aligned}\pi(f+g) &= \tau([f+g]) = \tau(W^*f+W^*g) = \tau(W^*W_1[f]+W^*W_2[g]) \\ &\leqq \|W^*W_1\|\tau([f]) + \|W^*W_2\|\tau([g]) \leqq \pi(f)+\pi(g).\end{aligned}$$

LEMMA 4. *If $f\in H$ and $a\in A$ then $\pi(fa)\leqq\|f\|\,|a|$.*

PROOF.

$$\pi(fa) = \tau([fa]) = \tau(W^*fa) \leqq |W^*f|\,|a| \leqq \|W^*\|\,\|f\|\,|a| \leqq \|f\|\,|a|.$$

Thus we see that the module operation is continuous in both variables.

LEMMA 5. *$\|f\| \leqq \pi(f)$ for all $f \in \tau(H)$.*

PROOF.

$$\|f\|^2 = \mathrm{tr}(f,f) = \tau([f]^2) \leqq \|L_{[f]}\|\tau([f]) \leqq |[f]|\tau([f]) = \|f\|\pi(f).$$

Our proof of completeness of $\tau(H)$ is based on a proof of McCarthy for a slightly different situation (Lemma 3.1 of [2]).

LEMMA 6. *Let $\{f_n\}$ be a sequence of elements of $\tau(H)$ and $f \in H$ with the following properties:*
*(i) $\{f_n\}$ is Cauchy in the $\pi(\ )$ norm.*
*(ii) $\|f_n - f\| \to 0$.*
*Then $f \in \tau(H)$ and $\pi(f) \leqq \liminf \pi(f_n)$.*

PROOF. Let $T_f$ be the map $a \to fa$. Then $T_f \in R(AH)$ and $\|T_f\| \leqq \|f\|$. Hence $\|T_{f_n} - T_f\| \leqq \|f_n - f\| \to 0$. For each $n$, let $S_n = (T_{f_n}^* T_{f_n})^{1/4}$ and $S = (T_f^* T_f)^{1/4}$. Then we have $\|S_n - S\| \to 0$. By our remark after Lemma 1, we know that $(T_{f_n}^* T_{f_n})^{1/2} = L_{[f_n]}$ for each $n$. Now $[f_n] \in \tau(A)$ implies that $S_n = L_{c_n}$ for some $c_n^2 \in A$ $(c_n^2 = [f_n])$. We want to show that $S = L_c$ for some $c \in A$. This will mean that $c^2 = [f]$, and hence $f \in \tau(H)$.

Consider an OPB $\{e_\alpha\}$ for $A$, and let $\sigma$ be a finite set of indices. Since $|c_n|^2 = \mathrm{tr}(c_n^2) = \pi(f_n)$ we see that $\{|c_n|^2\}$ is a bounded sequence. Then $\sum_{\alpha\in\sigma} |Se_\alpha|^2 = \lim_{n\to\infty} \sum_{\alpha\in\sigma} |c_n e_\alpha|^2$ and, for fixed $n$, $\sum_{\alpha\in\sigma} |c_n e_\alpha|^2 \leqq \sum_\alpha |c_n e_\alpha|^2 = |c_n|^2$. This implies that $\sum_{\alpha\in\sigma} |Se_\alpha|^2 \leqq \liminf_n |c_n|^2 < +\infty$. Since $\sigma$ is arbitrary, we have $\sum_\alpha |Se_\alpha|^2 \leqq \liminf_n |c_n|^2$. Setting $c = \sum_\alpha Se_\alpha$ we see that $c \in A$ and $S = L_c$. Moreover $\pi(f) = |c|^2 \leqq \liminf \pi(f_n)$.

PROOF OF COMPLETENESS OF $\tau(H)$. Let $\{g_n\}$ be a Cauchy sequence in $\tau(H)$. Then by Lemma 5, $\{g_n\}$ is a Cauchy sequence in $H$ also, and thus there exists $g \in H$ such that $\|g_n - g\| \to 0$. For fixed $n$, let $f_m = g_n - g_m$ and $f = g_n - g$. Then $\{f_m\}$ and $f$ satisfy the conditions of Lemma 6. Hence for each $n$, $g_n - g \in \tau(H)$, and $\pi(g_n - g) \leqq \liminf_m (g_n - g_m)$. Since $\{g_n\}$ is Cauchy in the $\pi(\ )$ norm, we have $\lim_{n\to\infty} \liminf_m \pi(f_n - f_m) = 0$.

4. **Proof of Theorem 2.** We first show that for $f \in \tau(H)$, $p_f$ is a bounded linear functional on $C(HA)$ with $\|p_f\| \leqq \pi(f)$. This follows from the inequalities

$$\begin{aligned} |p_f(T)| &= |\mathrm{tr}(Tf)| \leqq \tau(Tf) = \tau(TW[f]) \\ &\leqq \|TW\|\tau([f]) \leqq \|T\|\tau([f]) = \|T\|\pi(f). \end{aligned}$$

Here we have used Lemma 5 and the partial isometry $W$ of Lemma 1.

To show that $\|p_f\| = \pi(f)$, we consider the bounded linear functional $p_{[f]}$ which maps $T$ into $\operatorname{tr}(T[f])$ $(T \in C(A))$. We first show that $\pi(f) \leqq \|p_{[f]}\|$ and then $\|p_{[f]}\| = \|p_f\|$. Let $\{e_k\}$ be a sequence of mutually orthogonal projections in $A$ and $\{\lambda_k\}$ be a sequence of positive numbers such that $[f] = \sum_k \lambda_k e_k$. Then $\pi(f) = \operatorname{tr}([f]) = \sum_k \lambda_k |e_k|^2$. For each $n$, let $c_n = \sum_{k=1}^n e_k$. Then $|\operatorname{tr}(L_{c_n}[f])| \to \pi(f)$ as $n \to \infty$, and for each $n$,

$$|\operatorname{tr}(L_{c_n}[f])| = |p_{[f]}(L_{c_n})| \leqq \|p_{[f]}\| \, \|L_{c_n}\| \leqq \|p_{[f]}\|.$$

Thus we have $\pi(f) \leqq \|p_{[f]}\|$.

Let $W$ be the partial isometry such that $W[f] = f$. Then for $T \in C(HA)$, we have $p_f(T) = p_{[f]}(TW)$ and for $S \in C(A)$ we have $p_{[f]}(S) = p_f(SW^*)$. This implies that $\|p_{[f]}\| = \|p_f\|$.

So far we have shown that the map $f \to p_f$ is an isometric isomorphism of $\tau(H)$ into the dual of $C(HA)$. In order to show that the map is onto, we shall need the following lemma.

LEMMA 7. *If $a \in A$ and $T \in R(HA)$ then $L_a T$ is of the form $L_f$ for some $f \in H$.*

PROOF. From Theorem 3 of [4], we know that a member $S$ of $R(HA)$ with range in $\tau(A)$ is of the form $L_f$ if there exists a constant $M$ such that $\tau(Sg) \leqq M\|g\|$ for all $g \in H$. This condition is satisfied since $\tau(L_a Tg) = \tau(aTg) \leqq |a| \; |Tg| \leqq |a| \; \|T\| \, \|g\|$ for all $g \in H$.

Now consider a bounded linear functional $p$ on $C(HA)$. Define $\hat{p}(g) = p(L_g)$ for $g \in H$. Then

$$|\hat{p}(g)| = |p(L_g)| \leqq \|p\| \, \|L_g\| \leqq \|p\| \, \|g\| \quad \text{for all } g \in H.$$

Thus $\hat{p}$ is a bounded linear functional on $H$, and so there exists $f \in H$ such that

$$p(L_g) = \hat{p}(g) = [g, f] = \operatorname{tr}(f, g) = \operatorname{tr}(L_g f) \quad \text{for all } g \in H.$$

We must show that $f \in \tau(H)$. Let $\{e_k\}$ be a sequence of mutually orthogonal projections in $A$ and $\{\lambda_k\}$ a sequence of positive numbers such that $[f] = \sum_k \lambda_k e_k$. Let $\{e_\alpha\}$ be an extension of $\{e_k\}$ to an OPB for $A$. We will show that $\sum_\alpha ([f]e_\alpha, e_\alpha)$ is finite. For each $n$, let $c_n = \sum_{k=1}^n e_k$. Then

$$\sum_{k=1}^{n} ([f]e_k, e_k) = |\operatorname{tr}(c_n[f])| = |\operatorname{tr}(L_{c_n} W^* f)| = |p(L_{c_n} W^*)| \leqq \|p\| \, \|L_{c_n}\| \, \|W^*\| \leqq \|p\|.$$

Here we have used the fact that $L_{c_n} W^*$ is of the form $L_g$ for some

$g \in H$. Thus we have $\sum_\alpha ([f]e_\alpha, e_\alpha) = \sum_k ([f]e_k, e_k) \leqq \|p\|$, and by Lemma 2, we must have $f \in \tau(H)$. Since we have shown that $p$ agrees with $p_f$ on a dense subset of $C(HA)$, we conclude that $p = p_f$.

This concludes our proof of Theorem 2. At this point it should be clear that the above proof is a natural modification of the techniques used in Saworotnow's paper [5]. A similar modification yields the proof of Theorem 3, which will be omitted.

## References

1. P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton, N. J., 1967. MR **34** #8178.

2. C. A. McCarthy, $c_p$, Israel J. Math. **5** (1967), 249–271. MR **37** #735.

3. R. Schatten, *Normed ideals of completely continuous operators*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Heft 27, Springer-Verlag, Berlin, 1960. MR **22** #9878.

4. P. P. Saworotnow, *A generalized Hilbert space*, Duke Math. J. **35** (1968), 191–197. MR **37** #3333.

5. ———, *Trace-class and centralizers of an H*-algebra*, Proc. Amer. Math. Soc. **26** (1970), 101–104.

6. P. P. Saworotnow and J. C. Friedell, *Trace-class for an arbitrary H*-algebra*, Proc. Amer. Math. Soc. **26** (1970), 95–100.

7. P. P. Saworotnow and G. R. Giellis, *Continuity and linearity of centralizers*, Proc. Amer. Math. Soc. (submitted).

Catholic University of America, Washington, D.C. 20017

U. S. Naval Academy, Annapolis, Maryland 21402