A LIMITATION THEOREM FOR
ABSOLUTE SUMMABILITY

GODFREY L. ISAACS

Abstract. Let $A(a)$ be of bounded variation over every finite interval of the nonnegative real axis, and let $\int e^{-a} dA(u)$ be summable $\left| C, k \right|$ for a given integer $k \geq 0$ and a given $s$ whose real part is negative. Then it is known that the function $R(k, w) = (1/\Gamma(k+1)) \int e^{-a} (u-w)^k dA(u)$ (which certainly exists in the $\left| C, k \right|$ sense by a well-known summability-factor theorem) satisfies $e^{-\zeta w} R(k, w) = o(1) \left| C, 0 \right|$ as $w \to \infty$. In this paper we extend the above result by showing that if the hypotheses are satisfied with $k$ fractional, then $e^{-\zeta w} R(k+\delta, w) = o(1) \left| C, 0 \right|$ for each $\delta > 0$ and that this is best possible in the sense that $\delta$ may not be replaced by 0.

1. Let $A(a)$ be of bounded variation over every finite interval of the nonnegative real axis. We write

$$F(a; x) = \int_a^x f(u) dA(u) = L + o(1) \quad (C, k)$$

(read: $F(a; x)$ is summable $(C, k)$ to the limit $L$, or $\int_a^x f(u) dA(u)$ exists in the $(C, k)$ sense and equals $L$) if

$$\Gamma(k + 1)x^{-k}F_k(a; x) = x^{-k} \int_a^x (x - u)^k f(u) dA(u) \to L$$

as $x \to \infty$. (Stieltjes integrals are to be taken in the Riemann sense.) If in addition $x^{-k}F_k(a; x)$ is of bounded variation over $[a, \infty)$ we shall write $\left| C, k \right|$ instead of $(C, k)$ in the notations above.

This paper is concerned with the $(C, k)$ and $\left| C, k \right|$ summability of

$$C(a) \quad (= C(0; x)) = \int_0^x e^{-a} dA(a)$$

and of

$$R(k', w; x) = 1/\Gamma(k' + 1) \int_w^x (u - w)^{k'} dA(a).$$

Presented to the Society, April 28, 1969; received by the editors October 7, 1970.
AMS 1970 subject classifications. Primary 40A10, 40F05, 40G05; Secondary 40D05, 40D15.
Key words and phrases. Laplace-Stieltjes integral, Cesàro summability, summable $\left| C, k \right|$. 

Copyright © 1971, American Mathematical Society

47
We shall write
\[ R(k', w) = \frac{1}{\Gamma(k' + 1)} \int_{w}^{\infty} (u - w)^{k'} dA(u) \]
so that \( R(k', w) \) exists in the \((C, k)\) sense iff (3) is summable \((C, k)\).

In virtue of [1, p. 300], if (2) is summable \((C, k)\) (or \(|C, k|\)) for some \(k \geq 0, \Re(s) < 0\), then \( R(k', w) \) exists in the \((C, k)\) (or \(|C, k|\)) sense for each \(w \geq 0, k' \geq 0\). We have now:

**Theorem A** [5, pp. 412–413]. If \( k = 0, 1, 2, \ldots \), and (2) is summable \(|C, k|\), where \( \Re(s) = \sigma < 0 \), then
\[ e^{-w^*w^{-k}R(k, w)} = o(1) \quad |C, 0| \]

The last phrase will mean that the function on the left, \( g(w) \), say, tends to 0 as \( w \to \infty \) and is of bounded variation over \([1, \infty)\), i.e.,
\[ \int_{1}^{w} dg(u) = -g(1) + o(1) \quad |C, 0| \]

We state now, writing \([k]\) for the largest integer \(\leq k\), and \((k)\) for \(k - [k]\):

**Theorem A’.** If \( k \) is positive and fractional, and if (2) is summable \(|C, k|\) for some \( s \) such that \( \sigma < 0 \), then
\[ e^{-w^*w^{-k}R(k, w)} = B(w) + (-1)^{[k]+1}w^{-k}T(w), \]
where \( B(w) = o(1) \quad |C, 0| \) and
\[ T(w) = \frac{1}{\Gamma((k))} \int_{w}^{w+1} (u - w)^{(k)-1}C_{[k]}(u)du, \]

\( C(u) \) being given by (2).

**Theorem A”.** Under the hypotheses of Theorem A’,
\[ e^{-w^*w^{-k}R(k + \delta, w)} = o(1) \quad |C, 0| \quad \text{for each } \delta > 0. \]

**Theorem A’”.** Under the hypotheses of Theorem A’, \( e^{-w^*w^{-k}R(k, w)} \) is not necessarily bounded, even with \( e^{-w^*} \) replaced by \( e^{-wX} \) with \( X \) as large as we please.

Theorem A’’ is the extension of Theorem A to the case \( k \) fractional, and Theorem A’’’ shows that Theorem A” is best possible in the sense that \( \delta \) may not be replaced by 0.

2. We shall prove the following slight generalization of Theorem A’:

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Theorem A'. If $C(w)$ is summable $|C, k - \delta|$, where $k$ is positive and fractional, and $\sigma < 0$, $0 \leq \delta < \langle \delta \rangle$, then

$$e^{-w^\delta}R(k, w) = B^{(1)}(w) + (-1)^{[k] + 1}w^{k - \delta}T(w)$$

where $B^{(1)}(w) = o(1)$ $|C, 0|$ and $T(w)$ is given by (5).

By [6], slightly modified, the $(C)$ versions of Theorems A' and A" (obtained by replacing $|C, \cdots|$ by $(C, \cdots)$) hold. Thus it is sufficient to prove A'* and A" with $'= o(1)'$ replaced by 'is summable'. We shall use (see [6, (25)-(31)]):

Lemma 1. If for a given $k \geq 0$ and $\sigma < 0$, $C(w)$ is summable $(C, [k] + 1)$, then $R(k, w)$ exists in the $(C, [k] + 1)$ sense and

$$R(k, w) = \sum_{v=0}^{[k]+1} b_v Q(k, v, w)$$

where

$$Q(k, v, w) = \int_w^\infty C_{[k]}(u)(u - w)^{k - v}e^{u^\delta}du,$$

the integrals being convergent, and the $b_v$'s being constants, with

$$b_{[k]+1} = (-1)^{[k]+1}/\Gamma(\langle \delta \rangle).$$

Theorems A' and A" will be deduced from

Theorem A**. Under the hypotheses of Theorem A', $e^{-w^\delta}w^{k - \delta}Q(k, v, w)$ is summable $|C, 0|$ if either (i) $0 < \delta < \langle \delta \rangle$, $v \leq [k] + 1$, or (ii) $\delta = 0$, $v \leq [k]$.

We shall require

Lemma 2. Let $w \geq 1$, $-\infty \leq a < b \leq \infty$, and $a < u < b$. If

$$F(w) = \int_a^b g(w, u)f(u)du \quad \text{and} \quad \int_1^\infty |dw\ g(w, u)| \leq g(u),$$

then

$$\int_1^\infty |dF(u)| \leq \int_a^b g(u)|f(u)|du,$$

the integrals over $(a, b)$ being supposed existent in the Lebesgue sense.

Proof. If $w_0 = 1 < w_1 < \cdots < w_m$ we have

$$\sum_{n=1}^m |F(w_n) - F(w_{n-1})| \leq \int_a^b |f(u)| \sum_{n=1}^m |g(w_n, u) - g(w_{n-1}, u)|,$$

and the sum on the right is $\leq g(u)$ by hypothesis.
Proof of Theorem A**. We write
\[ p(t) = t^{k-1}C_{k-1}(t) \quad (t > 0), \]
\[ = 0 \quad (t = 0). \]
Then \( p(t) \) is of bounded variation over \([0, \infty)\). Let
\[ D(u, w) = C_{[k]+1}(u) - C_{[k]+1}(w). \]
Then integrating by parts in (6) and using \( C_{[k]+1}(u) = O(u^{[k]+1}) \), we have
\[ w^{k-1}e^{-w}Q(k, v, w) = (v - k)I_v - sI_{v-1} \]
where
\[ I_v = w^{k-1} \int_w^\infty (u - w)^{k-1}e^{-(u-w)}D(u, w)du. \]
Now \( \Gamma(\delta - (k)+1)D(u, w) \) can be expressed as
\[ \int_0^u (u - t)^{\delta-(k)}C_{k-1}(t)dt - \int_0^w (w - t)^{\delta-(k)}C_{k-1}(t)dt. \]
We write the first integral as the sum of integrals over \([w, u]\) and \([0, w]\) and then combine the second of these with the second integral in (11), thus obtaining \( X + Y \), say. We replace \( C_{k-1}(t) \) by \( t^{k-1}p(t) \) in each of these, and then put \( t = w + (u-w)y \) in \( X \) and \( t = w - x \) in \( Y \). Inserting the resultant expression in (10) and putting \( u = w + z \), we obtain for \( \Gamma(\delta - (k)+1)I_v \):
\[ \int_0^\infty z^{\delta - [1]}e^{zr}dz \int_0^1 (1 - y)^{\delta-(k)}r(z, y, w)p(w + zy)dy \]
\[ - \int_0^\infty z^{\delta-v-1}e^{zr}dz \int_0^w \{ x^{\delta-(k)} - (x + z)^{\delta-(k)} \} (1 - x/w)^{k-d}p(w - x)dx \]
\[ = L(w) - M(w), \]
say, where \( r(z, y, w) = (1+zy/w)^{k-\delta} \). Since \( r \) decreases as \( w \) increases,
\[ \int_1^\infty |d_w(rp(w + zy))| \leq \int_1^\infty |p(\partial r/\partial w)| \, dw \]
\[ + \int_1^\infty r |d_w p(w + zy)| \leq c(1 + z)^{k-\delta}, \]
say, where \( c \) is independent of \( z \) and \( y \). Hence by Lemma 2,
\[ \int_1^\infty |dL(w)| \leq c \int_0^\infty (1 + z)^{k-\delta}z^{\delta-[1]}e^{zr}dz \int_0^1 (1 - y)^{\delta-(k)}dy, \]
which is finite in either case (i) or case (ii). Next, let
\[ q(w, x) = (1 - x/w)^k \rho(w - x) \quad (0 \leq x < w), \]
\[ = 0 \quad (x \geq w). \]
Then
\[ \int_1^\infty |d_\omega q(w, x)| \leq \int_x^\infty |d_\omega ((1 - x/w)^k \rho(w - x))| \leq c', \]
where \( c' \) is independent of \( x \), by an argument similar to (13). Hence by Lemma 2,
\[ \int_1^\infty |dM(w)| \leq c' \int_0^\infty e^{p-2\omega-1}[e^{\omega-2\omega}(w) - (x + z)^{k-(k)}]dx, \]
which is finite in either case (i) or case (ii). Since, finally, each of these cases is satisfied by \( v-1 \) if it is satisfied by \( v \), the proof of Theorem A** is complete.

**Proof of Theorem A**. Put \( k - \delta = k' \) in Theorem A**, case (i). Then by Lemma 1 the function \( S^{(i)}(w) = e^{-w^2}e^{\omega-k}R(k', \omega, w) \) is of bounded variation over \([1, \infty)\) for each sufficiently small \( \delta > 0 \). Now by [6, Lemma 2] we have, for \( p > 0 \),
\[ e^{\omega-k}S^{(i+p)}(w) = 1 / \Gamma(p) \int_0^\infty (u - w)e^{-u}e^{\omega-k}(u)du. \]
The substitution \( u = w + x \) followed by an application of (our) Lemma 2 and an argument like that of (13) gives \( S^{(i+p)}(w) \) of bounded variation over \([1, \infty)\). This completes the proof.

**Proof of Theorem A**. By either case (i) or case (ii) of Theorem A**, together with Lemma 1, we have for \( 0 \leq \delta < k \),
\[ w^{k-2}e^{-w}R(k, w) = H(w) + w^{k-2}e^{-w}b_{k+1}Q(k, [k] + 1, w), \]
where \( H(w) \) is of bounded variation over \([1, \infty)\). We write, by (9),
\[ Q(k, [k] + 1, w) = \left( \int_w^{w+1} + \int_{w+1}^\infty \right) (u - w)^{(k-1)}e^{\omega}(\partial D/\partial u)du \]
\[ = J + K, \]
\[ J = e^{\omega-1} \int_w^{w+1} (u - w)^{(k-1)}C_{k+1}(u)du \]
\[ + \int_w^{w+1} (u - w)^{(k-1)}(e^{\omega} - e^{\omega-1}) \frac{\partial D}{\partial u}du \]
\[ = J_1 + J_2. \]
Integrations by parts of $K$ and $J_2$, followed by arguments along the lines of $(11)-(14)$, show that $e^{-\omega w^{k-k}}(K+J_2)$ is of bounded variation over $[1, \infty)$. By (7) this completes the proof.

Proof of Theorem $A''$. We shall use

**Lemma 3.** Suppose that $k$ is positive and fractional and that $y_n (n=1,2,\cdots)$ is a given sequence of positive numbers tending monotonically to $\infty$. Then there exists a function $C(u)$ such that

\begin{enumerate}
    \item[(a)] $C(u)$ is absolutely continuous over every finite interval of the nonnegative real axis, and $C(0)=0$;
    \item[(b)] $C(u)=O(1)$ with $C, k$; \\
\end{enumerate}

but such that the function $T(w)$ given by (5) satisfies $-T(2n) \geq c'y_n (n =1,2,\cdots)$ ($c'$ a positive constant).

**Proof.** Let $b, c$ satisfy $0<c-b<2$. We define (compare [3, p. 286]) a function $g_{b,c}(x)$ with domain $b \leq x \leq c$, such that it is symmetric about $x=(b+c)/2$ and

\[
g_{b,c}(x) = \left(1 - E^{[k]+2}\right)^{[k]+2} \quad (b \leq x \leq (b+c)/2),
\]

where $E=(b+c-2x)/(c-b)$. By induction on $r$, $g_{b,c}^{(r)}(x)$ has a factor $(1-E^{[k]+2})^{[k]+2}E^{[k]+2-r}$ for $b \leq x \leq (b+c)/2$ ($r=1,2,\cdots,[k]+1$), and thus $g_{b,c}^{(r)}(x)$ is 0 at $x=b, c, (b+c)/2$. The latter (with $x=b, c$) is clearly true also for $r=0$. For $r=[k]+2$ the function exists and is bounded in $b<x<(b+c)/2$ and in $(b+c)/2<x<c$. Further,

\[
\int_b^{(b+c)/2} |g_{b,c}(x)| \, dx = \int_{(b+c)/2}^c |g_{b,c}(x)| \, dx = 1.
\]

We now write $h_n = \frac{1}{2} e^{-\pi n}$, and define $G(u)$ as follows: for $0 \leq u \leq 1$, $G(u)=0$; and for $u \geq 1$ we have, taking $n=1,2,\cdots$,

\[
\begin{align*}
G(u) &= 0 \quad (2n \leq u < 2n+1), \\
&= 1/n \quad (2n-1+h_n \leq u < 2n-h_n), \\
&= (1/n)g_{2n-1,2n-1+2h_n}(u) \quad (2n-1 \leq u < 2n-1+h_n), \\
&= (1/n)g_{2n-2h_n,2n}(u) \quad (2n-h_n \leq u < 2n).
\end{align*}
\]

Then $0 \leq G(u) \leq 1$ for all $u>0$. We see that $G$ has a $[k]+1$th derivative everywhere, and a $[k]+2$th derivative almost everywhere, which is bounded on every finite interval. Hence we may choose $C(u)$ such that $C_k(u)=G(u)$, $C(0)=0$, and $C(u)$ is absolutely continuous over
every finite interval of the nonnegative real axis. Now by differentiating on the left side we have

\[ \int_1^\infty \left| \frac{d}{du} (u^{-k} C_k(u)) \right| du \leq 1 + \sum_{n=1}^{\infty} \left( \int_{2n-1}^{2n} + \int_{2n}^{2n+1} \right) u^{-k} \left| C_k(u) \right| du. \]

The second integral on the right is 0; and by (17) and (16) the first is \( \leq (2n-1)^{-k} n^{-1} (1+1) \), so that the sum is finite. Hence (15) is established. We now write, by (5),

\[ -\Gamma(\langle k \rangle) \Gamma(1 - \langle k \rangle) T(2n) \]

\[ = - \int_{2n}^{2n+1} (u - 2n)^{(k)-1} du \int_0^u (u - t)^{-(k)} C_k(t) dt. \]

We call this expression I. Replacing \( u \) by \( 2n \) in the inner integral (since \( C_k(t) = 0 \) for \( 2n \leq u \leq 2n+1 \)), then integrating the latter by parts, and thereafter using the fact that the resulting integral is decreased by replacing its limits by \( 2n - 1 + h_n \) and \( 2n - h_n \), we obtain, after an inversion,

\[ I \geq \langle k \rangle \int_p^q C_k(t) dt \int_{2n}^{2n+1} (u - 2n)^{(k)-1} (u - t)^{-(k)-1} du, \]

where \( p = 2n - 1 + h_n, q = 2n - h_n \). Writing \( u - t \) as

\[ \left( 1 - \frac{2n + 1 - u}{2n + 1 - t} \right) (2n + 1 - t), \]

expanding the \((\langle k \rangle - 1)\)th power of the first factor in a binomial series and then integrating term by term, we see that the last inner integral is \( \langle k \rangle^{-1} (2n+1-t)^{-(k)} (2n-t)^{-1} \). Hence by (17),

\[ I \geq n^{-1} \int_p^q (2n + 1 - t)^{-(k)} (2n - t)^{-1} dt \geq n^{-1} 2^{-(k)} \log \frac{2n - p}{2n - q}. \]

The definitions of \( p, q, h_n \), now give \( I \geq 2^{-\langle k \rangle} y_n \), which completes the proof.

**Proof of Theorem A'''.** For the given \( s \), let \( y_n = e^{2n} e^{-2ns} \). Let \( C(u) \) satisfy the conditions of Lemma 3, with this \( y_n \).

Choosing \( A(u) = \int_0^u e^{tu} dC(t) \), we see that by Theorem A',

\[ R(k, 2n) = e^{2ns} (2n)^k B(2n) + (-1)^{\lceil k \rceil + 1} e^{2ns} T(2n), \]

where the term involving \( B(2n) \) tends to 0 as \( n \to \infty \). But then
\[ |R(k, 2n)| \geq c''e^{\alpha n} \] for all \( n \) large enough, where \( c'' \) is a positive constant. This completes the proof.

In conclusion, I wish to thank Professor D. Borwein for his comments and for Lemma 2 and Theorem A**, which greatly reduced the complexity of my original proofs; also Professor W. H. J. Fuchs for his valuable suggestions.

Bibliography


Herbert H. Lehman College, City University of New York, Bronx, New York 10468