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EVERY COUNTABLE-CODIMENSIONAL SUBSPACE OF A BARRELLED SPACE IS BARRELLED[1]

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ABSTRACT. As indicated by the title, the main result of this paper is a straightforward generalization of the following two theorems by J. Dieudonné and by I. Amemiya and Y. Kōmura, respectively:

(i) Every finite-codimensional subspace of a barrelled space is barrelled.

(ii) Every countable-codimensional subspace of a metrizable barrelled space is barrelled.

The result strengthens two theorems by G. Köthe based on (i) and (ii), and provides examples of spaces satisfying the hypothesis of a theorem by S. Saxon.

Introduction. N. Bourbaki [**2**] observed that if E is a separable, infinite-dimensional Banach space, then E contains a dense subspace M of countably infinite codimension which is a Baire space. R. E. Edwards [**4**] noted that since M is Baire, it is an example of a non-complete normed space which is barrelled. Obviously, (i) and (ii) provide a plethora of such examples. It is apparently unknown whether every countable- (or even finite-) codimensional subspace of an arbitrary Baire space is Baire; (for closed subspaces the results are affirmative). In the second paper [**8**], which follows, the authors give topological properties other than "barrelledness" which are inherited by subspaces having the algebraic property of countable-codimensionality.

1. The notation will be that used by J. Horváth [**5**]. If (E, F) is a dual pairing (E and F not necessarily separating points) then $\sigma(E, F)$

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[1] It has just been learned that M. Valdivia has an independent proof of this fact, to appear in Ann. Inst. Fourier (Grenoble) under the title *Absolutely convex sets in barrelled spaces*.

[2] Some of the contributions of the first-named author are included in his Florida State University dissertation which was written under the supervision of Professor C. W. McArthur.

will denote the topology on E of pointwise convergence on F. $\tau(F, E)$ will denote the topology on F of uniform convergence on $\sigma(E, F)$ compact subsets of E. The vector space of continuous linear functionals on a locally convex space E will be designated as E'. A (not necessarily Hausdorff) locally convex space is said to be *barrelled* if every closed, balanced, convex, absorbing subset of it is a neighborhood of 0. A locally convex space E is said to be a *Mackey space* if it has the topology $\tau(E, E')$; to have *property* (S) if E' is $\sigma(E', E)$-sequentially complete. The *codimension* of a linear subspace M of a vector space E is the (algebraic) dimension of the quotient vector space E/M. ω will stand for the set of natural numbers.

2. To attack first the case of closed linear subspaces of countable codimension we prove the following:

LEMMA. *Let M be a closed linear subspace of countable codimension in a locally convex space E. If E has property* (S), *then any linear extension to E of a continuous linear functional on M is continuous.*

PROOF. Let $P=\{x_n : n\in\omega\}$ be a countable set such that E is the linear span of $M\cup P$. Let M_n be the linear subspace spanned by $M\cup\{x_1, \cdots, x_n\}$. Let f be a continuous linear functional on M and let f_0 be any linear extension of f on E. For each n, M is a closed linear subspace of finite codimension in M_n. Therefore the restriction f_n of f_0 to M_n is continuous. Let g_n be any continuous extension of f_n to E. An element x of E belongs to M_n for some $n\in\omega$. Therefore the sequence $\{g_n(x)\}_{n\in\omega}$ is eventually constant and converges to $f_0(x)$. Since E was supposed to have property (S), f_0 is a member of E', and is continuous.

PROPOSITION. *Let E be a Mackey space with property* (S). *Let M be a closed linear subspace of countable codimension in E. Let N be any algebraic supplement of M. Then N has the strongest locally convex topology and is a topological supplement of M.*

PROOF. Let π be the projection of E onto N along M. Let V be any absorbing, balanced, convex subset of N. Let $\mathfrak{J}$ be the given Mackey topology on E, and let $\mathfrak{J}_0$ be that locally convex topology on E which has as a fundamental system of neighborhoods of zero all sets of the form $U\cap\epsilon\cdot(M+V)=U\cap(M+\epsilon\cdot V)$, where U is a $\mathfrak{J}$-neighborhood of zero and $\epsilon>0$. Clearly, $\mathfrak{J}_0$ is stronger than $\mathfrak{J}$, so that $(E, \mathfrak{J})'\subset(E, \mathfrak{J}_0)'$. The relative topologies $\mathfrak{J}|_M$ and $\mathfrak{J}_0|_M$ are coincident on M. Thus $f\in(E, \mathfrak{J}_0)'\Rightarrow f|_M\in(M, \mathfrak{J}_0|_M)'=(M, \mathfrak{J}|_M)'\Rightarrow f\in(E, \mathfrak{J})'$ by the lemma. That is, $(E, \mathfrak{J})'=(E, \mathfrak{J}_0)'$. But since $\mathfrak{J}$ is the Mackey topology, we now have that $\mathfrak{J}$ is stronger than $\mathfrak{J}_0$, and $M+V$ is a $\mathfrak{J}$-neighborhood of zero. We have shown two things, then, that N has the strongest locally convex topology and that for any absorbing, balanced, convex subset V of N, $\pi^{-1}(V)=M+V$ is a neighborhood of zero in E. That is, π is continuous, and N is a topological supplement to M.

COROLLARY A. *A closed subspace M of countable codimension in a barrelled space E is barrelled.*

PROOF. A barrelled space is Mackey and has property (S), so the preceding proposition applies. M is isomorphic to E/N, so M is barrelled.

COROLLARY B. *A countable-dimensional barrelled space must be isomorphic to the product of a vector space with the strongest locally convex topology and one with the indiscrete topology.*

PROOF. The closure of $\{0\}$ is a closed subspace of countable codimension.

COROLLARY C. *A metrizable locally convex space E of countably infinite dimension is not barrelled.*

PROOF. If E were barrelled, E would have the strongest locally convex topology, since $\{0\}$ is closed. Then every linear functional on E would be continuous. E must contain a linearly independent sequence convergent to zero. The linear functional, necessarily continuous, which assumes value 1 on every element of the sequence provides a contradiction.

3. In handling the case of dense linear subspaces of countable codimension, we utilize the following:

LEMMA. *Let E be a locally convex space with property* (S). *Let A be a balanced, convex, closed subset of E. If the linear span* $\mathrm{sp}(A)$ *of A has countable codimension in E, then* $\mathrm{sp}(A)$ *is closed in E.*

PROOF. The proof is explicitly given for the $\aleph_0$-codimensional case. Let x_1 be any element of E not in $\mathrm{sp}(A)$, and let $\{x_k\}_{k=1}^{\infty}$ be a linearly independent sequence in E such that $\mathrm{sp}(\{x_k\})$ and $\mathrm{sp}(A)$ are algebraic supplements to each other. By the bipolar theorem [**5**, p. 192] and since E is locally convex, we have $A^{00}=A$, where the (absolute) polars are taken with respect to the canonical pairing between E and E'. Since no nonzero scalar multiple of x_k is in A, it thus follows that A^0 is unbounded at x_k for $k=1, 2, \cdots$. Let $\epsilon>0$ be given, with $\epsilon<1$. Let $A_1=A$ and let $A_n=A_{n-1}+\{tx_{n-1}: |t| \leqq 1\}$ for $n=2, 3, \cdots$. Then $A_n^{00}=A_n$ for $n=1, 2, \cdots$. Choose a sequence $\{f_n\}$ such that f_n

$\in(2^{-n}\cdot\epsilon)A_n^0$, $f_1(x_1)=2$, and $\sum_{k=1}^{n+1} f_k(x_{n+1})=0$ for $n=1, 2, \cdots$. This is possible since A_n^0 is balanced and $\{f(x_n): f\in A_n^0\}$ is unbounded for $n=1, 2, \cdots$. Let $x\in E$. Since A_n is absorbing in $\mathrm{sp}(A_n)$ and since $\bigcup_{n=1}^\infty \mathrm{sp}(A_n)=E$, there is some integer p and a positive number δ such that $\delta x\in A_p\subset A_{p+1}\subset A_{p+2}\cdots$. Thus

$$\sum_{k=1}^\infty |f_k(x)| = 1/\delta\sum_{k=1}^\infty |f_k(\delta x)| = 1/\delta\left(\sum_{k=1}^{p-1}|f_k(\delta x)| + \sum_{k=p}^\infty |f_k(\delta x)|\right)$$

$$\leqq 1/\delta\left(\sum_{k=1}^{p-1}|f_k(\delta x)| + \sum_{k=p}^\infty 2^{-k}\cdot\epsilon\right) < \infty.$$

Since E' is $\sigma(E', E)$-sequentially complete, the linear functional f_ϵ defined by $f_\epsilon(x)=\sum_{k=1}^\infty f_k(x)$ is continuous on E. Furthermore, for $x\in A_1$,

$$|f_\epsilon(x)| \leqq \sum_{k=1}^\infty |f_k(x)| \leqq \sum_{k=1}^\infty 2^{-k}\cdot\epsilon = \epsilon,$$

and

$$|f_\epsilon(x_{n+1})| \leqq \left|\sum_{k=1}^{n+1} f_k(x_{n+1})\right| + \sum_{k=n+2}^\infty |f_k(x_{n+1})| \leqq 0 + \sum_{k=n+2}^\infty 2^{-k}\cdot\epsilon < \epsilon$$

for $n=1, 2, \cdots$. Also, $|f_\epsilon(x_1)|\geqq 2-\sum_{k=2}^\infty |f_k(x_1)| > 2-\epsilon > 1$. Let $g_\epsilon=(f_\epsilon(x_1))^{-1}f_\epsilon$. Then $g_\epsilon(x_1)=1$, $|g_\epsilon(x)|<\epsilon$ for $x\in A_1=A$, and $|g_\epsilon(x_{n+1})|<\epsilon$ for $n=1, 2, \cdots$. Let $\{\epsilon_n\}$ be a sequence such that $0<\epsilon_n<1$ $(n=1, 2, \cdots)$ and $\epsilon_n\to 0$. The linear functional h defined on E by

$$\begin{aligned} h(x) &= 0 \qquad \text{if } x\in A_1,\\ &= 1 \qquad \text{if } x = x_1,\\ &= 0 \qquad \text{if } x = x_{n+1}, \quad \text{for } n = 1, 2, \cdots, \end{aligned}$$

is the pointwise limit of a sequence $\{g_{\epsilon_n}\}_{n=1}^\infty$ of continuous linear functionals, and hence is continuous. Therefore the closure of $\mathrm{sp}(A)$ is a subset of $h^{-1}(\{0\})$ and does not contain x_1. It follows that $\mathrm{sp}(A)$ is closed.

MAIN THEOREM. *Let M be a countable-codimensional linear subspace of a barrelled space E. Then M is barrelled.*

PROOF. Let T be a barrel in M. Then its closure $\overline{T}$ in E is a barrel in its linear span $\mathrm{sp}(\overline{T})$. Since $M=\mathrm{sp}(T)\subset\mathrm{sp}(\overline{T})\subset E$, the codimension of $\mathrm{sp}(\overline{T})$ in E is countable. Hence by the lemma,

$$\mathrm{sp}(\overline{T}) = \overline{\mathrm{sp}(T)} = \overline{M},$$

and $\overline{T}$ is a neighborhood of zero in the barrelled space $\overline{M}$, by Corollary A. It follows that $T=\overline{T}\cap M$ is a neighborhood of zero in M.

4. The following improves Köthe's extensions [7] of a perturbation theorem by T. Kato:

PROPOSITION. *Let E be a Pták space, F a barrelled space, and A a closed linear map on $D[A]\subset E$ into F. If the image space $R[A]$ is of countable codimension in F, then A is open and $R[A]$ is closed in F. Moreover, if F does not contain a complemented $\aleph_0$-dimensional subspace with the strongest locally convex topology (in particular, if F is metrizable) then $R[A]$ has finite codimension in F.*

PROOF. $R[A]$ is a barrelled space, so that the proof [7] of Köthe's first theorem applies, as does, in turn, our Proposition 2 (and its Corollary C).

Techniques used in S. Saxon's dissertation [9, Chapter 3] show, without use of the continuum hypothesis, that every subspace with codimension less than $2^{\aleph_0}$ in a Fréchet space is barrelled, and if closed, is finite-codimensional. This gives another generalization (the best possible as regards codimensionality) of Kato's theorem [6]. (It is apparently also new in Kato's original setting of Banach spaces.)

THEOREM. *Let E be a Pták space, F a Fréchet space, and A a closed linear map on $D[A]\subset E$ into F. If the codimension of $R[A]$ is less than $2^{\aleph_0}$, then A is open and $R[A]$ is closed and of finite codimension in F.*

5. The results of this paper (as well as [1], [3], and [4]) assure the existence of barrelled (and proper) subspaces of the Banach space l_1 which contain the unit vector basis $\{e_i\}$ of l_1. The importance of these spaces is evident in the following:

THEOREM. *Let K be the positive cone of a Schauder basis $\{x_i\}$ in a locally convex Hausdorff space E with $K-K$ barrelled. Then the following are equivalent.*

(i) *$\{x_i\}$ is a normalizable absolute (Schauder) basis.*

(ii) *E is, by identification, a barrelled subspace of l_1 which contains $\{e_i\}$ as a normalization of $\{x_i\}$.*

(iii) *K has a bounded base.*

(iv) *K has a closed and bounded base.*

This theorem is Corollary 7 to Theorem 2.6 of S. Saxon [9].

6. The following examples show that sometimes dense linear subspaces of uncountable codimension in barrelled spaces are barrelled, and sometimes they are not.

EXAMPLE A. *A dense linear subspace of uncountable codimension in a barrelled space which is barrelled.* The linear span L of a basis in a Banach space E is of countable dimension. By Corollary C to the proposition of paragraph 2, L is not barrelled. By the Main Theorem, L is of uncountable codimension. There must exist an increasing sequence $\{L_n\}_{n\in\omega}$ of linear subspaces of E such that L_n is of uncountable codimension in L_{n+1} for each $n\in\omega$, $L_1=L$ and $E=\bigcup_{n\in\omega} L_n$. Since E is a Baire space, it cannot be the union of a countable collection of sets which are nowhere dense in themselves. Then some L_n is a Baire space, and so is barrelled (cf. [2]).

EXAMPLE B. *A dense linear subspace of a barrelled space which has uncountable dimension and fails to be barrelled.* Let s be the space of real sequences with the product topology. Let m be the dense linear subspace of s consisting of the bounded sequences. m is not barrelled with the relative topology since the barrel $\{\{x_n\}_{n\in\omega}: |x_n| \leqq 1$ for $n\in\omega\}$ is not a neighborhood of 0 in the relativised topology. By the Main Theorem, m is of uncountable codimension.

REMARK. A similar example may be obtained in l_2 by considering the subspace spanned by the Hilbert cube.

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