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TAME ARCS ON WILD CELLS

CHARLES L. SEEBECK III[1]

ABSTRACT. We prove here that, for $n \geqq 5$, every cell in E^n contains a tame arc and that, for product cells $B^{m-k} \times I^k \subset E^{n-k} \times E^k = E^n$, every k-dimensional polyhedron $P \subset B^{m-k} \times I^k$ is tame in E^n.

1. **Introduction.** Bing showed in [1] that every 2-cell in 3-dimensional Euclidean space contains a tame arc and in [2] that there is a 2-sphere that is wild but for which all subarcs are tame. We obtain here analogous results in higher dimensions ($\geqq 5$). First we show that for $n \geqq 5$, any subarc of any k-cell in E^n can be approximated by subarcs tame in E^n. Then we show that if C is any $(m-k)$-cell in E^{n-k}, $I^k \subset E^k$ is the k-fold product of the unit interval I, $m \leqq n-2$, and $n \geqq 5$, then every sub k-cell of $C \times I^k \subset E^{n-k} \times E^k$ is tame in E^n. Since there are cells in this class of factored cells that are wild at every point [10] we have a generalization of Bing's example [2] to higher dimensions.

2. **The approximation theorems.** First we give a few definitions. Let $X \subset M$ be closed subsets of E^n. Let d denote the usual metric on E^n. A homeomorphism h of M is an ϵ-push of (M, X) if there is an isotopy h_t of M such that $h_0 = \text{Identity}$, $h_1 = h$, $d(h_t(x), x) < \epsilon$ for each $t \in I$ and each $x \in M$, and h_t is the identity outside the ϵ-neighborhood of X in M for each t. If P is a polyhedron and $h: P \to E^n$ is an embedding we say that h is tame if there is a homeomorphism H of E^n such that $H \cdot h$ is piecewise linear (PL).

LEMMA 1. *Suppose X is a compact subset of E^n,* Int $X = \varnothing$, *X does not locally separate E^n, G is a compact 1-dimensional subpolyhedron of E^n, $n \geqq 4$, and $\epsilon > 0$. Then there is an ϵ-push h of $(E^n, G \cap X)$ such that $h(G) \cap X = \varnothing$.*

PROOF. The proof is an immediate consequence of general position and Corollary 5.6 of [3].

LEMMA 2. *Suppose X is a compact subset of E^n,* Int $X = \varnothing$, *X does not locally separate E^n, P is a 2-dimensional subpolyhedron of E^n, $n \geqq 4$,*

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and $\epsilon>0$. *Then there is an* ϵ*-push* h *of* $(E^n, P\cap X)$ *such that* $h(P)\cap X$ *is totally disconnected.*

PROOF. Let K be a triangulation of P and $\{K_i \mid i=1, 2, \cdots\}$ the sequence of ith derived barycentric subdivisions of K. We shall use Lemma 1 to construct an ϵ-push h of $(E^n, P\cap X)$ such that $h(\bigcup |k_i^1|)\cap X=\emptyset$. Clearly h then satisfies the conclusion of Lemma 2.

Let $\epsilon_1=\epsilon/2$ and apply Lemma 1 with (X, G, ϵ) replaced by $(X, |K_1^1|, \epsilon_1)$, obtaining an ϵ_1-push h_1 of $(E^n, |K_1^1|\cap X)$ such that $h_1(|K_1^1|)\cap X=\emptyset$. Let $\delta_1=d(h_1(|K_1^1|), X)$ and η_1 be some positive number chosen depending on h_1. (See [8] or Theorem 3.4 of [7].) Set $\epsilon_2=\min\{\epsilon_1/2, \delta_1/2, \eta_1\}$. As before we obtain an ϵ_2-push h_2' of $(E^n, h_1|K_2^1|\cap X)$ such that $h_2'\cdot h_1(|K_2^1|)\cap X=\emptyset$. Set $h_2=h_2'\cdot h_1$. Continuing in this way we obtain a sequence $\{h_i\}$ of homeomorphisms of E^n. Since $\epsilon_{i+1}<\epsilon_i/2$, $\lim_{i\to\infty} h_i=h$ is an ϵ-map of E^n supported on a compact set. Because $\epsilon_{i+j}\leqq\delta_i/2^j$, $h(\bigcup|K_i^1|)\cap X=\emptyset$ and because the η_i are chosen sufficiently small depending on the h_i, h is a homeomorphism. Thus h is the required ϵ-push of (E^n, X).

THEOREM 3. *Let* M *be a PL* m*-manifold topologically embedded in* E^n, G *a* 1*-dimensional subpolyhedron of* M, G' *a subpolyhedron of* G *that is tame in* E^n, $n\geqq 5$, *and* $m\geqq 2$. *Then for each* $\epsilon>0$ *there is an* ϵ*-embedding* $\alpha: G\to M$ *such that* α *is tame in* E^n *and* $\alpha|G'=$ *inclusion*: $G'\to E^n$.

PROOF. It is sufficient to consider the case that M is a 2-cell, G is an arc, and $G'=\{\text{end points of } G\}$. Using Lemma 2 we shall construct an embedding $\alpha: G\to M$ fixed on G' such that $E^n-\alpha(G)$ is uniformly locally 1-connected (1-ULC). By Theorem 4.2 of [4], α is thus tame.

Let K be a triangulation of E^n and $K_1, K_2, \cdots$ the sequence of ith derived barycentric subdivisions of K. It follows from general position and Lemma 2 that there is an $\epsilon/2$-push h_1 of $(E^n, G\cap|K_1^2|)$ such that $h_1(|K_1^2|)\cap M$ is totally disconnected and $h_1(|K_1^2|)\cap G'=\emptyset$. Thus there is an $\epsilon/2$-embedding $\alpha_1: G\to M$ fixed on G' such that $\alpha_1(G)\cap h_1|K_1^2|=\emptyset$. Now as in the proof of Lemma 2 we set $\epsilon_1=\epsilon/2$, $\delta_1=d(\alpha_1(G), h_1(|K_1^2|))$, $\eta_1>0$ chosen depending on α_1, $\eta_1'>0$ chosen depending on h_1, $\epsilon_2=\min\{\epsilon_1/2, \delta_1/4, \eta\}$, and $\epsilon_2'=\min\{\epsilon_1/2, \delta_1/4, \eta_1'\}$. Then using Lemma 2 we find an ϵ_2'-push h_2' of $(E^n, h_1(|K_2^2|)\cap M)$ such that $h_2'\cdot h_1(|K_2^2|)\cap M$ is totally disconnected. Let $h_2=h_2'\cdot h_1$. We can again find an embedding $\alpha_2: G\to M$ such that $d(\alpha_1, \alpha_2)<\epsilon_2$, $\alpha_2(G)\cap h_2(|K_2^2|)=\emptyset$, and α_2 is fixed on G'. Continuing in this way we construct a sequence $\alpha_i: G\to M$ of ϵ_i-embeddings and a sequence $\{h_i\}$ of homeomorphisms of E^n. Because $\epsilon_{i+1}\leqq\epsilon_i/2$, the α_i converge to an ϵ-map $\alpha: G\to M$. The η_i can be picked so as to guarantee that α is an

ϵ-embedding. Similarly the h_i converge to an ϵ-push of (E^n, M). Because $\max\{\epsilon_{i+j}, \epsilon'_{i+j}\} \leqq \delta_i/2^{j+1}$, $\alpha(G)\cap h(\bigcup_{i=1}^{\infty}|K_i^2|)=\emptyset$. Thus $h^{-1}\cdot\alpha(G)\cap\bigcup_{i=1}^{\infty}|K_i^2|=\emptyset$, and so $E^n-h^{-1}\cdot\alpha(G)$ is 1-ULC. Therefore $h^{-1}\cdot\alpha$ and hence α is tame.

COROLLARY 3.1. *Suppose N is a PL n-manifold, M is a PL m-manifold topologically embedded in N, G is a 1-dimensional polyhedron, $G'\subset G$ is a subpolyhedron, $\beta:G\rightarrow M$ is an embedding such that $\beta|G':G'\rightarrow N$ is tame, $n\geqq 5$, and $m\geqq 2$. Then for each $\epsilon>0$ there is an embedding $\alpha:G\rightarrow M$ such that $d(\alpha,\beta)<\epsilon$, $\alpha|G'=\beta|G'$, and $\alpha:G\rightarrow N$ is tame.*

PROOF. First take an infinite triangulation of $G-G'$ and approximate β by an embedding $\beta':G\rightarrow M$ such that $\beta'|G'=\beta|G'$ and β' is locally PL on $G-G'$. Then apply Theorem 3 to a sequence of compact subpolyhedra of $\beta'(G-G')$. Thus we obtain an embedding $\alpha:G\rightarrow M$ such that $\alpha|G'=\beta|G'$ and $\alpha|G-G'$ is locally tame in E^n. Thus $\alpha:G\rightarrow N$ is tame (Theorem 4.2 of [4]).

THEOREM 4. *Suppose N is a PL n-manifold, M is a PL m-manifold topologically embedded in N, every 2-complex of M can be approximated by a 2-complex in M that is tame in N, and $5\leqq m\leqq n-2$. Then each k-dimensional polyhedron P topologically embedded in M, $k<m$, can be approximated in M by embeddings that are tame in N.*

PROOF. It follows from [5] and either [6] or [9] that an approximation of P is tame if its complement is 1-ULC. Such an approximation is found by modifying the proof of Theorem 3. Let L be a triangulation of M and $L_1, L_2, \cdots$ the sequence of barycentric subdivisions. Similarly let $K_1, K_2, \cdots$ be the sequence of barycentric subdivisions of a triangulation of N. Using techniques similar to those above it is possible to construct a homeomorphism h of N such that $h(\bigcup|K_i^1|)\cap M=\emptyset$ and $h(\bigcup|K_i^2|)\cap(\bigcup Q_i)=\emptyset$ where Q_i is a close approximation of $|L_i^2|$ for each i that is tame in N. Now for each i we can find an arc $A_i\subset M$ such that $C_i=h(|K_i^2|)\cap M\subset A_i$ and A_i-C_i is locally tame in M. Since $M-C_i$ is 1-ULC, $M-A_i$ is 1-ULC and hence A_i is tame. Using A_i we can construct, for each i, a homeomorphism f_i of M moving points a distance depending on f_{i-1} so that $f_i(P)\cap C_i=\emptyset$. Thus as in the proof of Lemma 2 and Theorem 3 we can construct an ϵ-push f of (M, P) such that $f(P)\cap(\bigcup C_i)=\emptyset$. Thus $N-f(P)$ is 1-ULC and the required approximation has been found.

It is evident that we have actually proved the following.

ADDENDUM TO THEOREM 4. *Under the hypotheses of Theorem 4 it is possible to find for each $\epsilon>0$ an ϵ-push f of (M, P) such that $f|P:P\rightarrow N$ is tame.*

3. **Subpolyhedra of factored cells.** We say that an m-cell $C\subset E^n$ factors k-times if for some homeomorphism $h:E^n\to E^n$ and some $(m-k)$-cell $B\subset E^{n-k}$, $h(C)=B\times I^k\subset E^{n-k}\times E^k$ where I^k is the k-fold product of the interval I naturally embedded in E^k and $B\times I^k\subset E^{n-k}\times E^k$ is the product embedding.

THEOREM 5. *Suppose C is an m-cell topologically embedded in E^n, C factors k-times, $n\geqq 5$, and $m\leqq n-2$. Then every embedding of any compact k-dimensional polyhedron into C is tame in E^n.*

PROOF. Let B be an $(m-k)$-cell in E^{n-k}, P a finite k-dimensional polyhedron topologically embedded in $B\times I^k\subset E^{n-k}\times E^k$, $n\geqq 5$, and $1\leqq k<m\leqq n-2$. It follows from [5] and either [6] or [9] that P is tame in E^n if E^n-P is 1-ULC. However, E^n-P is 1-ULC if each 2-complex in E^n can be homotopied off P by arbitrarily small homotopies. Let K be a finite 2-complex. First find a very small homotopy of $|K|$ such that for some subdivision K' each 2-cell of K' either projects onto a 0- or 1-simplex of E^{n-k} or else lies in $E^{n-k}\times t$ for some $t\in E^k$. Since $n-k\geqq(m-k)+2E^{n-k}-B$ is locally 0-connected. Thus it follows that any 0- or 1-simplex in E^{n-k} can be homotopied off B by a small homotopy. Thus any 2-cell of K' that projects onto a 0- or 1-cell of E^{n-k} can be homotopied off $B\times I^k$. Let σ be a 2-cell of K', $t\in E^k$, and $\sigma\subset E^{n-k}\times t$. For $n-k\geqq 4$ it follows from Lemma 2 that there is an ϵ-push h of $(E^{n-k}\times t, \sigma)$ such that $h(\sigma)\cap(B\times t)$ is 0-dimensional. For $n-k=3$ we can use the techniques of the proof of Lemma 2 to find an embedding $h:\sigma\to E^{n-k}\times t$ such that $h(\sigma)\cap(B\times t)$ is 0-dimensional and h is close to the inclusion of σ into $E^{n-k}\times t$. Let $A=h(\sigma)\cap(B\times I^k)$. A is a 0-dimensional subset of $B\times t$. Let P be a k-dimensional polyhedron topologically embedded into $B\times I^k$. Let $T\subset P$ be defined as follows: $x\in T$ if there is a neighborhood U of x in P and a point $y\in E^{n-k}$ such that $U\subset y\times I^k$. Then T is open in P and P is locally tame at each point of U [5]. We shall construct a map $f:B\times E^k\to B\times E^k$ such that $p_1\cdot f=p_1$ where $p_1=$projection: $B\times E^k\to B$, $f(A)\cap P\subset T$, and $d(f, \mathrm{Id}|B\times E^k)$ is small. For each $(x, t)\in A\cap(P-T)$, let $\epsilon_x>0$ be chosen so that for some $t_x\in E^k$ with $d(t_x, t)<\epsilon_x$ and for all $x'\in B$ with $d(x', x)<\epsilon_x$, $(x', t_x)\in(B\times E^k)-P$. Now for some finite number of $x\in B$, the ϵ_x-neighborhoods of the x's cover $p_1(A\cap(P-T))$. Since A is totally disconnected it is possible to cover $p_1(A\cap(P-T))$ by closed sets $B_1, \cdots, B_k$ that are pairwise disjoint and, for each $i=1, \cdots, k$, there is an x_i such that B_i lies in the ϵ_{x_i}-neighborhood of x_i. Define $f(x,y)=(x,y+t_{x_i}-t)$ for $x\in B_i$. Then extend $p_2\cdot f:\bigcup B_i\times E^k\to E^k$ to a map $f_2:B\times E^k\to E^k$ such that $d(f_2, p_2)<\epsilon$. Then extend f to $B\times E^k$ by setting $f=\mathrm{Id}\times f_2:B\times E^k\to B\times E^k$. Then $f(A)\cap P\subset T$. Now f can be extended to an ϵ-map of E^n such that

$p_1 \cdot f = p_1 : E^n \to E^{n-k}$. Thus $f \cdot h(\sigma) \cap P \subset T$. Since P is locally tame at each point of T there is an approximation g of $f \cdot h$ such that $g(\sigma) \cap P = \emptyset$. Thus $E^n - P$ is 1-ULC and P is tame in E^n.

COROLLARY 5.1. *Let $C \subset E^n$ be an m-cell that factors* 1-*time. Let P be a k-dimensional polyhedron topologically embedded in C, $k < m \leqq n-2$, and $n \geqq 5$. Then for each $\epsilon > 0$ there is an ϵ-push H of (C, P) such that $H(P)$ is tame in E^n.*

PROOF. This is actually a corollary to the proofs of Theorem 3 and Theorem 5. Let K be a triangulation of E^n and suppose $C = B \times I \subset E^{n-1} \times E^1$. Then there is an approximation j of the inclusion map i: $|K^2| \to E^n$ such that $j(|K^2|) \cap C$ is a 0-dimensional subset of $B \times \{t_1, \cdots, t_p\}$ for some numbers $t_1, \cdots, t_p \in I$. Thus for any k-dimensional polyhedron $P \subset C$, there is a small homeomorphism h of C such that $h(P) \cap j|K^2| = \emptyset$. Thus we can obtain by a sequence of such steps a small homeomorphism H of C such that $E^n - H(P)$ is 1-ULC. Thus $H(P)$ is tame.

REMARKS. Do Theorem 3 and Theorem 5 remain true if the hypothesis $n \geqq 5$ is replaced by $n = 4$? Does Theorem 5 remain true if the hypothesis $m \leqq n-2$ is replaced by $m = n-1$? More specifically take Bing's 2-sphere $S \subset E^3$ [2]. Are all subarcs of $S \times I \subset E^4$ tame?

Theorem 5 is sharp in the sense that there are examples of cells that factor k-times and for which some $(k+1)$-dimensional subcell is wild.

Daverman has independently proved Theorem 3 for the case $m = 2$.

REFERENCES

1. R. H. Bing, *Each disk in E^3 contains a tame arc*, Amer. J. Math. **84** (1962), 583–590. MR **26** #4331.

2. ———, *A wild surface each of whose arcs is tame*, Duke Math. J. **28** (1961), 1–15. MR **23** #A630.

3. R. H. Bing and J. M. Kister, *Taming complexes in hyperplanes*, Duke Math. J. **31** (1964), 491–511. MR **29** #1626.

4. J. L. Bryant and C. L. Seebeck III, *Locally nice embeddings of polyhedra*, Quart. J. Math. Oxford Ser. (2) **19** (1968), 257–274. MR **38** #2751.

5. ———, *Locally nice embeddings in codimension three*, Quart. J. Math. Oxford Ser. (2) **21** (1970), 265–272.

6. A. V. Černavskiĭ, *Topological embeddings of manifolds*, Dokl. Akad. Nauk SSSR **187** (1969), 1247–1250=Soviet Math. Dokl. **10** (1969), 1037–1041.

7. H. Gluck, *Embeddings in the trivial range*, Ann. of Math (2) **81** (1965), 195–210. MR **30** #3456.

8. T. Homma, *On the embedding of polyhedra in manifolds*, Yokohama Math. J. **10** (1962), 5–10. MR **27** #4236.

9. R. Miller, *Approximating codimension* 3 *embeddings* (to appear).

10. T. B. Rushing, *Everywhere wild cells and spheres* (to appear).

MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48823