

MODULUS SPACE IS SIMPLY-CONNECTED

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ABSTRACT. Modulus space of genus g consists of conformal equivalence classes of compact Riemann surfaces of genus g . It is the quotient space of the real Teichmüller space of marked Riemann surfaces by the Teichmüller modular group. The result follows using generators and some relations of this group and the fact that it acts discontinuously on Teichmüller space.

1. Let D denote the unit disc $\{z \mid |z| < 1\}$ and \mathcal{L} the topological group of conformal homeomorphisms of D . A *Fuchsian group* is a discrete subgroup of \mathcal{L} . If S is a compact Riemann surface of genus $g \geq 2$, then S is conformally equivalent to D/K_g where K_g is a Fuchsian group with presentation:

Generators: $a_1, b_1, a_2, b_2, \dots, a_g, b_g$

Relation: $\prod_{j=1}^g a_j b_j a_j^{-1} b_j^{-1} = 1.$

Let K_g denote a fixed Fuchsian group of the above type and $R(K_g)$ the set of isomorphisms $\phi: K_g \rightarrow \mathcal{L}$. $R(K_g)$ can be topologised by identifying ϕ with the point $(\phi(a_1), \phi(b_1), \dots, \phi(a_g), \phi(b_g))$ of \mathcal{L}^{2g} . The group, \mathcal{L}^+ , of angle-preserving homeomorphisms of D , acts on $R(K_g)$ by $(\gamma(\phi))(k) = \gamma\phi(k)\gamma^{-1}$ for $\gamma \in \mathcal{L}^+$ and all $k \in K_g$. The quotient space, $T(K_g)$, is *Teichmüller space* of genus g and is homeomorphic to \mathcal{R}^{6g-6} . (See [2].)

Let $A(K_g)$ denote the automorphism group of K_g and $I(K_g)$, the inner automorphism group. The quotient $M(K_g) = A(K_g)/I(K_g)$ acts on $T(K_g)$ by

$$\bar{\alpha}[\phi] = [\phi \circ \alpha] \quad \text{where } \bar{\alpha} = \alpha I(K_g).$$

$M(K_g)$ is a *Teichmüller modular group* and is a properly discontinuous group of homeomorphisms of $T(K_g)$. (See [4].) $T(K_g)/M(K_g)$ is the *modulus space* of genus g , and is a space of all conformally equivalent compact Riemann surfaces of genus g .

2. Lickorish [5] has obtained generators for $M(K_g)$ and Birman [3] some relations between these generators. In proving Theorem 1

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below we use the methods and notation of [3]. $M(K_\theta)$ is generated by

$$\{h_{u_i}, h_{z_i}, h_{c_j}; 1 \leq i \leq g; 1 \leq j \leq g-1\}.$$

THEOREM 1. $M(K_\theta)$ can be generated by elements of finite order.

PROOF. In [3], it is shown that $\alpha = h_{c_1}h_{u_1}h_{z_1}h_{u_2}h_{z_2} \cdots h_{u_g}h_{z_g}$ has finite order $2g+2$. Using the same methods, one can show that $\beta = h_{u_1}h_{z_1}h_{u_2}h_{z_2} \cdots h_{u_g}h_{z_g}$ has finite order $4g+2$. Let N be the normal subgroup generated by all conjugates of α and β so that N is generated by elements of finite order. Then $M(K_\theta)/N$ will be generated by

$$\{\bar{h}_{u_i}, \bar{h}_{z_i}, \bar{h}_{c_j}; 1 \leq i \leq g, 1 \leq j \leq g-1\}$$

which satisfy all the relations of $M(K_\theta)$ together with the additional relations $\bar{\alpha} = 1, \bar{\beta} = 1$. But these imply that $\bar{h}_{c_1} = 1$. Repeated application of relations (7), (8) and (9) of [3], then gives $\bar{h}_{c_j} = \bar{h}_{u_i} = \bar{h}_{z_i} = 1$ for all i, j . Thus $M(K_\theta) = N$.

THEOREM 2. Modulus space is simply-connected.

PROOF. It is proved in [4] that each element of finite order in $M(K_\theta)$ has a fixed point in $T(K_\theta)$, so that, by Theorem 1, $M(K_\theta)$ is generated by elements which have fixed points. Also $M(K_\theta)$ is a properly discontinuous group of homeomorphisms of a space homeomorphic to \mathbf{R}^{6g-6} . Furthermore, the stabiliser of a point $[\phi]$ of $T(K_\theta)$ is isomorphic to the group of conformal self-homeomorphisms of the compact Riemann surface $D/\phi(K_\theta)$ and hence is finite. Thus, applying a result of Armstrong [1] we have that $T(K_\theta)/M(K_\theta)$ has trivial fundamental group.

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