PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 29, Number 1, June 1971

IDEMPOTENT MEASURES ON LOCALLY COMPACT SEMIGROUPS

A. MUKHERJEA AND N. A. TSERPES

ABSTRACT. A conjecture that the support of an r^*-invariant regular finite measure on a locally compact semigroup is a left group is proven. Moreover we also prove that the support of an idempotent measure on a locally compact semigroup is completely simple, thus extending a well-known result of Pym and Heble-Rosenblatt on compact semigroups to the locally compact case. These results are also shown to be true in a complete metric semigroup.

1. **Introduction.** Throughout S will be a locally compact Hausdorff topological semigroup and μ a regular (Borel) probability measure on the Borel σ-algebra $\mathcal{B}$ (generated by the open sets) of S. The support of μ will be denoted by $F=\{x;\ \mu(V)>0$ for every open $V\ni x\}$. If μ, ν are any two regular probability measures, the convolution $\mu*\nu$ is defined as the regular probability measure on S generated by the linear functional on $C(S)$ (= the space of all continuous real functions on S with compact support) defined by

$$L(f) = \int\left[\int f(xy)\mu(dx)\right]\nu(dy) \qquad ([3, \text{p. } 177] \text{ and } [11, \text{p. } 19]).$$

The measure μ is said to be indempotent if $\mu*\mu=\mu$. It can be shown that for $B\in\mathcal{B}$ [3, p. 180],

$$\mu*\nu(B) = \int \mu(Bx^{-1})\nu(dx)$$
$$= \int \mu(dx)\nu(x^{-1}B), \quad \text{where} \quad Bx^{-1} = \{s;\, sx \in B\}.$$

(Similarly one defines $x^{-1}B$.) μ is said to be r^*-invariant, whenever for every $B\in\mathcal{B}$ and $x\in S$, $\mu(B)=\mu(Bx^{-1})$. If μ is idempotent, F is a (closed) semigroup such that $\mathrm{cl}(FF)=F$ (see [5] and [4]). For $a\in F$, we denote by $\mu_a\equiv\mu(\cdot a^{-1})$ the measure $\mu_a(B)\equiv\mu(Ba^{-1})$, $B\in\mathcal{B}$. (Similarly for ${}_a\mu(B)=\mu(a^{-1}B)$.) μ_a is also a regular probability measure ([3, p. 179], [4]) and has support $\mathrm{cl}(Fa)$. The function of x, $\mu(Bx^{-1})$,

Received by the editors March 13, 1970 and, in revised form, July 31, 1970.

AMS 1969 *subject classifications*. Primary 6008, 2875.

Key words and phrases. Locally compact topological semigroup, regular Borel measure, idempotent measure, completely simple semigroup, right invariant measure.

for fixed $B \in \mathcal{B}$, is (Borel) measurable. In fact, for fixed open $V \subset S$, $V_\alpha \equiv \{s \in S;\ \mu(Vs^{-1}) > \alpha\}$ is open so that the function $\mu(Vx^{-1})$ is lower semicontinuous [3, p. 179]. A semigroup S is called completely simple if S is simple and contains a primitive idempotent [2, p. 46]. S is called a left group if S is left simple ($Sx = S$ for all $x \in S$) and contains an idempotent [1].

It has been conjectured [1] that if μ is r^*-invariant, then F is a left group. From [1] it follows that if one could prove this conjecture, then a complete characterization of r^*-invariant measures will be obtained. In this paper we give a proof of this conjecture. Also we prove that if μ is idempotent, then F is completely simple. This was proved for compact S in [3] and [5] and under various compactness conditions in [4] and [6] and conjectured in [6]. As in the compact case, the result that F is completely simple enables us to formulate a complete characterization of idempotent measures on locally compact semigroups as products of a Haar measure and two regular Borel measures (see Theorem 3.1).

2. LEMMA 2.1. *If μ is r^*-invariant, then for every $a \in F$, aF is left cancellative.*

PROOF. For every $x \in F$, $\mathrm{cl}(Fx) = F$ since μ is r^*-invariant [1]. Let $(ab)(ac) = (ab)(ad)$. Then (since $\mathrm{cl}(Fab) = F$), $s(ac) = s(ad)$ for all $s \in F$. Again, since $a \in \mathrm{cl}(Fa)$, $ac = ad$. (For if $s_\beta a$ is a net converging to a, then $s_\beta ac = s_\beta ad$ and hence $ac = ad$.)

The measure μ is called l^*-invariant if $\mu(x^{-1}B) = \mu(B)$ for every Borel $B \subset S$ and every $x \in F$.

LEMMA 2.2. *If μ is l^*-invariant, then Fa is right cancellative for every $a \in F$.*

PROOF. As in Lemma 2.1, we note that $\mathrm{cl}(xF) = F$ for every $x \in F$. Then from $(ca)(ba) = (da)(ba)$ if follows (as in Lemma 2.1) that $ca = da$.

LEMMA 2.3. *If μ is both r^*-invariant and l^*-invariant, then F is a compact group.*

PROOF. We first note that for every $a \in F$, $aF \cap Fa$ is bicancellative by Lemmata 2.1 and 2.2. Now let K be any compact set ($\subset F$) such that $\mu(K) > .99$. We observe that $aKa \subset aF \cap Fa$ and

$$\mu(aKa) = \mu(a^{-1}(aKa)a^{-1}) \geqq \mu(K) \geqq .99.$$

We consider the product space $(aKa \times aKa,\ \mu \times \mu)$. We extend $\mu \times \mu$ to a unique regular measure m_2 on the Borel subsets of $aKa \times aKa$.

(See H. Royden, *Real analysis*, Macmillan, New York, 1963, p. 314.) Following Gelbaum and Kalisch (*Measures in semigroups*, Canad. J. Math. **4** (1952), 396–406) we define the mappings θ and π on $S\times S$ by $\theta(x, y) = (x, yx)$ and $\pi(x, y) = (y, x)$. Then π is measure preserving on the product σ-field of $aKa\times aKa$. Since θ is continuous, $\theta(aKa\times aKa)$ is compact and hence m_2-measurable. Let U be open in the relative topology of $aKa\times aKa$ such that $\theta(aKa\times aKa) \cap aKa\times aKa = D\subset U$ and $m_2(U) < m_2(D) + .01$. Then there is a compact G_δ-set H such that $D\subset H\subset U$. Since H, being a compact G_δ, is $\mu\times\mu$-measurable, we have

$$m_2(D) \geqq m_2(H) - .01 = \mu\times\mu(H) - .01 = \int_{aKa} \mu(H_x)\mu(dx) - .01$$
$$> .95 \times .99 - .01 \geqq .8,$$

since $\mu(aKax) \geqq \mu(aKa) \geqq .99$ and since $\mu(H_x) \geqq \mu(D_x) = \mu(aKax\cap aKa) > .95$. ($D_x$ = the section of D by x; $[\theta(aKa\times aKa)]_x = aKax$.) It follows that $\pi(D)\cap D \neq \varnothing$. Hence there are x, y, u, v in aKa such that $(x, yx) = (vu, u)$ so that $x = vu$, $yx = u$ and $(vy)(vy)x = (vy)x$; by cancellation in aKa, there is an idempotent e $(=vy)$ in $aF\cap Fa$. Since Fe is closed and $Fe\subset Fa$, $Fa = \mathrm{cl}(Fe) = F$ and by dual arguments (using l^*-invariance), $aF = F$ for all $a\in F$, so that F is a group. Since F is now a locally compact semigroup which is algebraically a group, by [**2**, p. 36], F is a topological group and is compact since μ is a finite Haar measure on F. (Note that Fe, eF and eFe are all closed in F whenever e is an idempotent in F. For, if $x\in \mathrm{cl}(Fe)$, then there is a net $\{f_\beta e\}$ in Fe converging to x so that $x = xe \in Fe$ since F is a topological semigroup and $f_\beta e = f_\beta(ee)$ converges to xe and F is Hausdorff [**2**, p. 60].)

Remark. Since for Lemma 2.3, we only need that $\mu(Cx) \geqq \mu(C)$ and $\mu(xC) \geqq \mu(C)$ for every compact C and $x\in S$, this lemma proves the main result in [**8**] by a new proof independent of the method used in [**8**]. The main result of [**8**] is essentially Lemma 2.3. The proof presented above has the advantage of being applicable in the infinite case (demonstrated in a forthcoming paper of the authors in Semigroup Forum) and also applicable when S is a complete metric semigroup.

3. The importance of the conjecture that F is completely simple is shown by the following theorem (first given in [**6**]).

Theorem 3.1. *Let μ be idempotent and suppose that its support F is a (closed) completely simple semigroup. Then:*

(i) *For any* $e \in E(F) \equiv$ *the set of all idempotents of* F, $X \equiv E(Fe)$ *and* $Y \equiv E(eF)$ *are (closed) left-zero and right-zero semigroups in* F *respectively*; $G \equiv eFe$ *is a closed subgroup and* $YX \subset G$; *the product space* $X \times G \times Y$ *(product topology) is a locally compact semigroup and is isomorphic (both algebraically and topologically) to* F. *Multiplication on* $X \times G \times Y$ *is defined by* $(x_1, g_1, y_1)(x_2, g_2, y_2) \equiv (x_1, g_1(y_1x_2)g_2, y_2)$.

(ii) *The measure* μ *decomposes on* $X \times G \times Y$ *as a product measure* $\mu = \mu_X \times \mu_G \times \mu_Y$, *where* μ_X, μ_Y *are Borel probability measures on* X *and* Y *resp. and* μ_G *is the normed Haar measure on* G *which turns out be a compact group. Conversely any such product measure on* F *is idempotent on* S.

PROOF. (i) The representation $F = X \times G \times Y$ is always valid for any locally compact (Hausdorff) completely simple topological semigroup F by [**2**, pp. 49, 61, 62], [**10**] and [**6**]. (ii) The desired factorization of μ follows as in [**3**, p. 183] since the proof given there also applies to the locally compact case. G becomes compact since μ_G is a finite Haar measure on G. The converse case can be verified easily (see [**7**, p. 150]).

THEOREM 3.2. *If* μ *is idempotent on* S, *then* F *is completely simple.*

PROOF. We show first that for every $a \in F$, μ_a is r^*-invariant on its support $\mathrm{cl}(Fa)$. (The bars in the head below denote closure.)

To this end, we wish to show that $\mu(Va^{-1}x^{-1}) = \mu(Va^{-1})$ for every a and x in F and every open $V \subset F$. If $Va^{-1} = \varnothing$, then the equality is obvious. So let us assume that for $a \in F$, $\mu(Va^{-1}) = \alpha > 0$. For $0 < \epsilon < \alpha$, let $V_\alpha = \{x;\ \mu(Vx^{-1}) > \alpha - \epsilon\}$. Then V_α is nonempty and open since $\mu(Vx^{-1})$ is a lower semicontinuous function of x. By [**3**, p. 183], there exists a set E of zero μ_s-measure for every s in F such that $\mu(V_\alpha s^{-1}) = 1$ for every $s \in V_\alpha - E$. It follows that the support of μ_s is contained in $\overline{V}_\alpha$, i.e., $\mathrm{cl}(Fs) \subset \overline{V}_\alpha$, for every s in $V_\alpha - E$. Therefore $F\overline{V}_\alpha \subset \overline{V}_\alpha$, since $V_\alpha - E$ is dense[1] in $\overline{V}_\alpha$. Now let K be compact such that $K \subset V$ and $\mu(Ka^{-1}) \geqq \alpha - \epsilon$. By local compactness we can find an open set W such that $\overline{W}$ is compact and $K \subset W \subset \overline{W} \subset V$. Then if

$$W_\alpha = \{x;\ \mu(Wx^{-1}) > \alpha - 2\epsilon\},$$

then $W_\alpha \neq \varnothing$ and as above, $F\overline{W}_\alpha \subset \overline{W}_\alpha$. Since the function $\mu(\overline{W}x^{-1})$ is upper semicontinuous,

$$\overline{W}_\alpha \subset \{x;\ \mu(\overline{W}x^{-1}) \geqq \alpha - 2\epsilon\} \subset \{x;\ \mu(Vx^{-1}) \geqq \alpha - 2\epsilon\},$$

[1] If U is open containing $x \in \overline{V}_\alpha$, then $U \cap V_\alpha$ is a nonempty open set (hence having positive μ-measure) which intersects $F - E$, E being of μ-measure zero $[\mu(E) = \int \mu_s(F)\mu(ds)]$.

the set $\{x;\mu(\overline{W}x^{-1})\geqq\alpha-2\epsilon\}$ being closed; since $a\in W_\alpha$ it follows that for every $y\in F$, $ya\in\overline{W}_\alpha$ and $\mu(Va^{-1}y^{-1})\geqq\alpha-2\epsilon$. Therefore for every $y\in F$, $\mu(Va^{-1}y^{-1})\geqq\mu(Va^{-1})$, so that by idempotence of μ,

$$\mu(Va^{-1}y^{-1}) = \mu(Va^{-1}) \qquad \text{for almost all } y \in F.$$

By lower semicontinuity of $\mu(Vx^{-1})$, $\mu(Va^{-1}y^{-1})=\mu(Va^{-1})$ for all $y\in F$. Therefore $\mu(Va^{-1}y^{-1})=\mu(Va^{-1})$ for every a, $y\in F$ and every open V. If K is compact,

$$\mu(Ka^{-1}y^{-1}) = 1 - \mu(K^ca^{-1}y^{-1}) = 1 - \mu(K^ca^{-1}) = \mu(Ka^{-1}).$$

Next let B be any Borel set and $\epsilon>0$. Let K be compact and U be open such that $K\subset B\subset U$ and $\mu_a(U-K)<\epsilon$. Then

$$\begin{aligned}\mu(Ba^{-1}) &\leqq \mu(Ka^{-1}) + \epsilon = \mu(Ka^{-1}y^{-1}) + \epsilon \leqq \mu(Ba^{-1}y^{-1}) + \epsilon \\ &\leqq \mu(Ua^{-1}y^{-1}) + \epsilon = \mu(Ua^{-1}) + \epsilon \leqq \mu(Ba^{-1}) + 2\epsilon.\end{aligned}$$

Therefore $\mu(Ba^{-1}y^{-1})=\mu(Ba^{-1})$ for every a, $y\in F$ and every Borel B.

Consider for any $a\in F$, the measure μ_a on $\mathrm{cl}(Fa)$. We wish to show that μ_a is r^*-invariant on $\mathrm{cl}(Fa)$. For any $z\in F$,

$$\mu_a(B(za)^{-1}) = \mu(Ba^{-1}z^{-1}a^{-1}) = \mu(Ba^{-1}(az)^{-1}) = \mu_a(B);$$

also if $z\in\mathrm{cl}(Fa)$, and $z_\alpha a$ is a net converging to z, then for compact K,

$$\mu_a(Kz^{-1}) \geqq \limsup \mu_a[K(z_\alpha a)^{-1}] = \mu_a(K).$$

On the other hand, given $\epsilon>0$, if U is open, $U\supset K$ and $\mu_a(U-K)<\epsilon$, then

$$\mu_a(Kz^{-1}) \leqq \mu_a(Uz^{-1}) \leqq \liminf \mu_a(U(z_\alpha a)^{-1}) = \mu_a(U) \leqq \mu_a(K) + \epsilon.$$

(Note that $\mu_a(Ux^{-1})$ is a lower semicontinuous function of x.) Hence μ_a is r^*-invariant for every compact set $K\subset\mathrm{cl}(Fa)$. Next let B be any Borel set in $\mathrm{cl}(Fa)$; let $x\in\mathrm{cl}(Fa)$; since $\mu_{ax}=\mu_a(\cdot x^{-1})$ is also regular, we can find compact sets K_1, K_2 contained in B such that

$$\mu_a(Bx^{-1}) \leqq \mu_a(K_1x^{-1}) + \epsilon = \mu_a(K_1) + \epsilon \leqq \mu_a(B) + \epsilon$$

and

$$\mu_a(B) \leqq \mu_a(K_2) + \epsilon = \mu_a(K_2x^{-1}) + \epsilon \leqq \mu_a(Bx^{-1}) + \epsilon.$$

This proves that μ_a is r^*-invariant on $G\equiv\mathrm{cl}(Fa)$. Since [3, p. 179] $\mu(B)=\int\mu(Bx^{-1})\mu(dx)=\int\mu(x^{-1}B)\mu(dx)$ by dual arguments as above, it follows that ${}_a\mu\equiv\mu(a^{-1}B)$ is l^*-invariant (i.e. ${}_a\mu(x^{-1}B)={}_a\mu(B)$) on its support $H\equiv\mathrm{cl}(aF)$. We consider next the measure $m(B)\equiv\mu(a^{-1}Ba^{-1})$ $\equiv{}_a\mu_a(B)$ on its support $\mathrm{cl}(aFa)\equiv Q$. This measure is regular and both

l^*-invariant and r^*-invariant. Therefore it follows from Lemma 2.3 that the support Q of m is a compact group. Since G is the support of an r^*-invariant measure, $\mathrm{cl}(Gy) = G$ for all $y \in G$ [1]. Now Gy as a left ideal of G and Q as a right ideal of G intersect each other and hence $Q \subset Gy$ and e (= the identity of Q) is in Gy. It follows that $Ge \subset Gy$ and since Ge is closed, $Gy = G$ for all y and G is a left group and hence $G = \mathrm{cl}(Fa) = Fa$ is a minimal left ideal of F. Similarly, $H = \mathrm{cl}(aF) = aF$ is a right group and minimal right ideal. [The claims that $Q \subset Gy$ and $e \in Gy$ follow since $Q \subset G$, $Q \cap Gy$ is a nonempty left ideal of Q and therefore $Q \cap Gy = Q$, Q being a group. Note that $Q \cap Gy \supset (Q \cdot G)y$. Also $\mathrm{cl}(Fa) = \mathrm{cl}(Fa) \cdot fa \subset Fa$ if $f \in F$, $a \in F$, since G is a left group.] Now if K denotes the completely simple kernel of F, then $K \supset FF$ since Fa is a minimal left ideal of F for each a in F. Hence the kernel K is dense because $\mathrm{cl}(FF) = F$, μ being an idempotent measure. By [2, p. 61], K is closed since the mapping $\eta(x, g, y) = xgy$ is continuous in our case. Hence $K = F$.

THEOREM 3.3. *Let μ be an r^*-invariant probability measure on S. Then the support F of μ is a left group.*

PROOF. Since r^*-invariance implies idempotence for μ, by Theorem 3.2, F is completely simple and therefore $\mathrm{cl}(Fx) = Fx = F$ for every $x \in F$. [Note that Fx contains an idempotent e for every x in F and so $F = \mathrm{cl}(Fe) = Fe \subset Fx \subset F$.] Hence F is a left group.

We note that this theorem answers the conjecture in [1] in the affirmative for finite measures. The problem for infinite measures seems to be still open.

4. In this section we make the following remarks which will establish the validity of Theorems 3.2 and 3.3 and Lemma 2.3 when S is a complete metric semigroup.

REMARK 1. If μ is a regular Borel probability measure on S, S being a completely regular Hausdorff semigroup, the function $\mu(Bx^{-1})$, for fixed B in $\mathcal{B}$, is Borel measurable. For, consider $\mathcal{Q} = \{B \in \mathcal{B}; \mu(Bx^{-1})$ is measurable$\}$. Then $\mathcal{Q}$ contains all open sets U since $\mu(Ux^{-1})$ is lower semicontinuous by [3, p. 179] and $\mathcal{Q}$ is a monotone class. Since the open sets may not form an algebra, we consider the class of all sets of form $U \cap C$ (where U is open and C is closed) and their finite disjoint unions. This class is an algebra and is contained in $\mathcal{Q}$, since $\mu(U \cap C)x^{-1} = \mu(Ux^{-1}) - \mu(U \cap C^c)x^{-1}$. Hence $\mathcal{Q}$ is a σ-algebra containing all the Borel sets.

REMARK 2. If a regular probability measure μ is idempotent on a metric semigroup S, that is, $\mu(B) = \int \mu(Bx^{-1})\mu(dx)$ for every $B \in \mathcal{B}$, then F, the support of μ, is a closed separable metric semigroup since it contains a σ-compact set of measure 1, μ being regular. Then considering the product σ-algebra on $F\times F$, we see that for every Borel set B, the set $W=\{(x, y); xy\in B, x\in F, y\in F\}$ is product-measurable since it is so whenever B is an open set. Hence, by Fubini's theorem,

$$\mu(B) = \int_F \mu(Bx^{-1})\mu(dx) = \int_F \int_F I_B(yx)\mu(dy)\mu(dx)$$

$$= \int_F \int_F I_B(yx)\mu(dx)\mu(dy) = \int \mu(y^{-1}B)\mu(dy),$$

$I_B=$the characteristic function of B.

REMARK 3. If μ is as in Remark 2, then the proof of Theorem 3.2 shows that there exists a completely simple kernel K in F, which is dense in F. In [**2**, p. 62], local compactness is used to show that K is closed, where, however, the fact that $eFe=eKe$ is a topological group (continuous inversion in eFe) is needed and is sufficient for K to be closed. But when S is a complete metric semigroup, we note that eFe (where e is an idempotent in K) is a closed (hence complete) metric (separable) semigroup which is algebraically a group and hence a topological group. [See T. Husain, *Introduction to topological groups*, Saunders, Philadelphia, Pa., 1966, p. 38.]

REMARK 4. In the proof of Theorem 3.2 local compactness was used to find a set W such that $K\subset W\subset \overline{W}\subset V$ with $\overline{W}$ compact. Actually we only need $\overline{W}$ to be closed and such a set can be found by normality in a metric semigroup.

BIBLIOGRAPHY

1. L. N. Argabright, *A note on invariant integrals on locally compact semigroups*, Proc. Amer. Math. Soc. **17** (1966), 377–382. MR **32** #5780.

2. J. F. Berglund and K. H. Hofmann, *Compact semitopological semigroups and weakly almost periodic functions*, Lecture Notes in Math., no. 42, Springer-Verlag, Berlin and New York, 1967. MR **36** #6531.

3. M. Heble and M. Rosenblatt, *Idempotent measures on a compact topological semigroup*, Proc. Amer. Math. Soc. **14** (1963), 177–184. MR **30** #214.

4. A. Mukherjea, *Idempotent probabilities on semigroups*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **11** (1969), 142–146. MR **39** #3540.

5. J. S. Pym, *Idempotent measures on semigroups*, Pacific J. Math. **12** (1962), 685–698. MR **26** #6298.

6. T. C. Sun and N. A. Tserpes, *Idempotent measures on locally compact semigroups*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **15** (1970), 273–278.

7. A. Tortrat, *Lois tendues μ sur un demi-groupe topologique completement simple X*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **6** (1966), 145–160. MR **35** #1063.

8. N. A. Tserpes and A. Kartsatos, *Mesures semi-invariantes sur un semi-groupe* *localement compact*, C.R. Acad. Sci. Paris Sér. A–B **267** (1968), A507–A509. MR **38** #6032.

9. N. A. Tserpes and A. Mukherjea, *Some problems on idempotent measures on semigroups*, Bull. Austral. Math Soc. **2** (1970), 299–315.

10. A. D. Wallace, *The Rees-Suschkewitsch structure theorem for compact simple semigroups*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 430–432. MR **18**, 14.

11. J. H. Williamson, *Harmonic analysis on semigroups*, J. London Math. Soc. **42** (1967), 1–41. MR **34** #8101.

UNIVERSITY OF SOUTH FLORIDA, TAMPA, FLORIDA 33260