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COEFFICIENTS OF FUNCTIONS WITH BOUNDED BOUNDARY ROTATION

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ABSTRACT. For $k \geqq 2$ denote by V_k the class of normalized functions analytic in the unit disc which have boundary rotation at most $k\pi$. For fixed $n \leqq (k+6)/4$ we determine the maximum of the set of values of $|a_n|$, where a_n is the nth Taylor coefficient of a function in V_k.

For a fixed $k \geqq 2$ let V_k denote the class of functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \tag{1}$$

which are analytic in $U = \{z : |z| < 1\}$ and have an integral representation of the form

$$f'(z) = \exp\left\{\frac{1}{\pi}\int_0^{2\pi} \log(1 - ze^{-it})^{-1} d\mu(t)\right\}, \tag{2}$$

where $\mu(t)$ is real-valued and of bounded variation on $[0, 2\pi]$ with

$$\int_0^{2\pi} d\mu(t) = 2\pi, \qquad \int_0^{2\pi} |d\mu(t)| \leqq k\pi.$$

V. Paatero [7] showed that $f(z)$ given by (1) belongs to V_k if and only if $f'(z) \neq 0$ in U and $f(z)$ maps U onto a domain with boundary rotation at most $k\pi$. (See [4] for a definition of this concept.)

V_2 is precisely the class of normalized univalent functions that map U onto a convex domain, and it is known [7] that for $2 \leqq k \leqq 4$, V_k consists only of univalent functions. It is also known [1] that $f(z) \in V_k$ if and only if there exist $s_1(z)$ and $s_2(z) \in S^*$, the class of normalized starlike functions, such that

$$f'(z) = \frac{(s_1(z)/z)^{(k+2)/4}}{(s_2(z)/z)^{(k-2)/4}}. \tag{3}$$

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[1] The results in this paper are part of the author's Ph.D. thesis, written at the University of Maryland under the direction of Professor W. E. Kirwan. The author at present holds a National Research Council Postdoctoral Resident Research Associateship at the Naval Research Laboratory, Washington, D. C.

This follows directly from (2).

In spite of considerable effort, the problem of determining $M(n, k) = \max_{f \in V_k} |a_n|$ where $f(z)$ is given by (1) has been solved for all n only when $k=2$ and $k=4$. K. Löwner [6] proved that $M(n, 2)=1$, and A. Rényi [8] proved that $M(n, 4)=n$. O. Lehto [4] proved that $M(2, k)=k/2$ and $M(3, k)=(k^2+2)/6$. It is also known [2] that $M(4, k)=(k^3+8k)/24$. The extremal function in all of these cases has been

$$F_k(z) = \frac{1}{k}\left\{\left(\frac{1+z}{1-z}\right)^{k/2} - 1\right\}. \tag{4}$$

It is also known ([3], [9]) that $M(n, k) \leqq C(k) n^{k/2-1}$ where $C(k)$ depends only on k, and [4] that

$$M(n, k) \sim k^{n-1}/n! \qquad (k \to \infty). \tag{5}$$

Again $F_k(z)$ is extremal. These facts tend to support the conjecture that $F_k(z)$ is the solution of the coefficient problem $M(n, k)$ for all n and for all k. The purpose of this paper is to show that if $n \leqq [(k+6)/4]$, then $F_k(z)$ is the solution to the problem $M(n, k)$.

Before we can prove this, however, we need two lemmas. The proofs of these lemmas are modeled after that of Lemma 4 [5, p. 169], which in turn is due to Professor M. S. Robertson.

LEMMA 1. *Let $s(z) \in S^*$ and let $\beta > 0$. Let $g(z) = (s(z)/z)^{-\beta} = \sum_{n=0}^{\infty} b_n z^n$. Then for $0 \leqq n \leqq \beta+1$ we have*

$$|b_n| \leqq \frac{\Gamma(2\beta+1)}{\Gamma(n+1)\Gamma(2\beta-n+1)},$$

where Γ represents the gamma function.

PROOF. From the definition of $g(z)$ we have

$$\log g(z) = -\beta \log \frac{s(z)}{z}.$$

By differentiating both sides of this equation, we see that

$$1 - zg'(z)/\beta g(z) = zs'(z)/s(z).$$

Since $s(z) \in S^*$, we see that

$$P(z) = 1 - zg'(z)/\beta g(z)$$

belongs to $\mathcal{P}$, the class of normalized functions with positive real part. Therefore, $(P(z))^{-1} = 1 + \sum_{m=1}^{\infty} \mu_m z^m \in \mathcal{P}$, and $|\mu_m| \leqq 2$ for all m.

Now $(P(z))^{-1}(\beta g(z)-zg'(z))=\beta g(z)$. Thus by equating coefficients of z^n we see that

$$nb_n = \sum_{m=0}^{n-1} (\beta - m)b_m\mu_{n-m}.$$

A simple induction argument, in which we use the fact that $|\mu_{n-m}| \leqq 2$, now completes the proof. The requirement that $n \leqq \beta+1$ is used in the induction argument to insure that $\beta-(n-1) \geqq 0$.

LEMMA 2. *Let* $s(z) \in S^*$ *and let* $\alpha > 0$. *Let* $g(z)=(s(z)/z)^\alpha = \sum_{n=0}^\infty b_n z^n$. *Then*

$$|b_n| \leqq \frac{\Gamma(n+2\alpha)}{\Gamma(n+1)\Gamma(2\alpha)}.$$

PROOF. As in the proof of Lemma 1 we see that

$$\log g(z) = \alpha \log \frac{s(z)}{z},$$

which shows that

$$P(z) = 1 + \frac{zg'(z)}{\alpha g(z)} = 1 + \sum_{n=1}^\infty \delta_n z^n$$

belongs to $\mathcal{P}$. Thus $|\delta_n| \leqq 2$.

By expanding $g(z)$ and $g'(z)$ as power series and then comparing coefficients of z^n, we see that

$$nb_n = \alpha \sum_{m=0}^{n-1} b_m \delta_{n-m}.$$

The proof is then completed by induction. Note that this lemma is true for all n.

We are now able to prove the following theorem on the coefficients of a V_k function.

THEOREM. *Let* $f(z)=z+\sum_{n=2}^\infty a_n z^n \in V_k$, *and let* $F_k(z)=z+\sum_{n=2}^\infty A_n(k)z^n$ *be given by* (4). *Then for* $n \leqq [(k+6)/4]$ *we have* $|a_n| \leqq |A_n(k)|$ *with equality for any* $n \leqq [(k+6)/4]$ *if and only if* $f(z) = e^{i\theta}F_k(e^{-i\theta}z)$ *for some* $\theta \in [0, 2\pi]$.

PROOF. By (3) we have that

$$f'(z) = \frac{(s_1(z)/z)^{(k+2)/4}}{(s_2(z)/z)^{(k-2)/4}}$$

where $s_1(z)$, $s_2(z) \in S^*$. Let

$$(s_1(z)/z)^{(k+2)/4} = \sum_{j=0}^{\infty} c_j z^j$$

and

$$(s_2(z)/z)^{-(k-2)/4} = \sum_{j=0}^{\infty} b_j z^j.$$

Then

$$f'(z) = \sum_{m=1}^{\infty} m a_m z^{m-1} = \left(\sum_{j=0}^{\infty} c_j z^j\right)\left(\sum_{h=0}^{\infty} b_h z^h\right).$$

By equating coefficients we see that

$$(n+1)a_{n+1} = c_0 b_n + c_1 b_{n-1} + \cdots + c_{n-1} b_1 + c_n b_0.$$

From Lemma 2, with $\alpha = (k+2)/4$, we have that

$$|c_j| \leqq \frac{\Gamma(j + k/2 + 1)}{\Gamma(j+1)\Gamma(k/2+1)} \qquad (j \geqq 0)$$

and from Lemma 1, with $\beta = (k-2)/4$, we have that

$$|b_m| \leqq \frac{\Gamma(k/2)}{\Gamma(m+1)\Gamma(k/2 - m)} \qquad (0 \leqq m \leqq (k+2)/4).$$

Therefore, for $n \leqq [(k+2)/4]$ we have that

$$(n+1)|a_{n+1}| \leqq \sum_{j=0}^{n} \frac{\Gamma(j + k/2 + 1)}{\Gamma(j+1)\Gamma(k/2+1)} \cdot \frac{\Gamma(k/2)}{\Gamma(n-j+1)\Gamma(k/2 - (n-j))}. \tag{6}$$

Now let us examine

$$F_k'(z) = \frac{(1-z)^{-(k+2)/2}}{(1+z)^{-(k-2)/2}}.$$

Let $(1-z)^{-(k+2)/2} = \sum_{n=0}^{\infty} C_n z^n$ and $(1+z)^{(k-2)/2} = \sum_{m=0}^{\infty} B_m z^m$. Then

$$C_n = \frac{\Gamma(n + k/2 + 1)}{\Gamma(n+1)\Gamma(k/2+1)} \qquad (n \geqq 0)$$

and

$$B_n = \frac{\Gamma(k/2)}{\Gamma(n+1)\Gamma(k/2-n)} \qquad (0 \leqq n \leqq k/2).$$

Then we see that

$$(n+1)A_{n+1}(k) = C_0B_n + C_1B_{n-1} + \cdots + C_{n-1}B_1 + C_nB_0.$$

Since we require that $n \leqq [(k+2)/4]$, we have also that $n \leqq [k/2]$. Therefore

$$(n+1)A_{n+1}(k) = \sum_{j=0}^{n} \frac{\Gamma(j+k/2+1)}{\Gamma(j+1)\Gamma(k/2+1)} \cdot \frac{\Gamma(k/2)}{\Gamma(n-j+1)\Gamma(k/2-(n-j))} \qquad (n \leqq [(k+2)/4]). \tag{7}$$

Comparing (6) with (7), we find that

$$|a_{n+1}| \leqq |A_{n+1}(k)| \qquad (n \leqq [(k+2)/4]).$$

This is equivalent to the statement of the theorem.

If equality holds for any n, then an examination of the above proof shows that $|a_2| = A_2(k) = k/2$. But it is well-known [4] that $|a_2| = k/2$ if and only if $f(z) = e^{i\theta}F_k(e^{-i\theta}z)$ for some $\theta \in [0, 2\pi]$. This completes the proof of the theorem.

This theorem allows us to improve Lehto's result (5) in the following sense.

COROLLARY. *Let $F_k(z) = z + \sum_{n=2}^{\infty} A_n(k)z^n$ be given by* (4). *Let n be fixed but arbitrary, $n \geqq 2$. Then for $k \geqq 4n-6$ we have $M(n, k) = A_n(k)$. In particular, for any fixed n, $M(n, k) \sim k^{n-1}/n!$ $(k \to \infty)$.*

PROOF. Since $k \geqq 4n-6$, we have $n \leqq [(k+6)/4]$. Then, by the theorem, we see that $M(n, k) = A_n(k)$. Direct computation then shows that $M(n, k) \sim k^{n-1}/n!$ $(k \to \infty)$.

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