IRREDUCIBLE LIE ALGEBRAS OF INFINITE TYPE

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Abstract. Let $V$ be a finite dimensional vector space over an algebraically closed field of characteristic $\neq 2, 3, 5$. It is shown that if $L \leq \text{gl}(V)$ is an irreducible Lie algebra of infinite type then either $L = \text{gl}(V)$, $L = \text{sl}(V)$, $\dim V = 2r \geq 4$ and $L = \text{sp}(V)$, $\dim V = 2r \geq 4$ and $L = \text{csp}(V)$, or there exists $A \in L$ such that $\text{ad} A \neq 0 = (\text{ad} A)^2$. As a corollary we obtain E. Cartan's classification of the irreducible Lie algebras of infinite type over $\mathbb{C}$.

Let $L$ be a Lie algebra of linear transformations of a vector space $V$. For each nonnegative integer $n$ the $n$th Cartan prolongation, $L_n$, is defined inductively by $L_0 = L$ and

$$L_n = \{ \phi \in \text{Hom}(V, L_{n-1}) \mid y(x\phi) = x(y\phi) \text{ for all } x, y \in V \}$$

for $n \geq 1$. If $L_n \neq (0)$ for all $n \geq 0$ then $L$ is said to be of infinite type. The main result of this paper is:

**Theorem 1.** Let $V$ be a finite dimensional vector space over an algebraically closed field of characteristic $\neq 2, 3, 5$. If $L$ is an irreducible Lie algebra of infinite type then either $L = \text{gl}(V)$, $L = \text{sl}(V)$, $\dim V = 2r \geq 4$ and $L = \text{sp}(V)$, $\dim V = 2r \geq 4$ and $L = \text{csp}(V)$, or there exists an $A \in L$ such that $\text{ad} A \neq 0 = (\text{ad} A)^2$.

Now it is easily seen (as in [7]) that if $(\text{ad} A)^2 = 0$ then $A$ belongs to the radical of the Killing form. If $\Phi = \mathbb{C}$ and $L$ is reductive this implies that $\text{ad} A = 0$. Furthermore, it is known (Theorem 1 of [4]) that an irreducible Lie algebra over $\mathbb{C}$ is of infinite type if and only if $L_2 \neq (0)$. Thus Theorem 1 implies the following theorem of E. Cartan [1]:

**Theorem 2.** Let $V$ be a finite dimensional vector space over $\mathbb{C}$ and let $L$ be an irreducible Lie algebra of linear transformations of $V$ such that $L_2 \neq (0)$. Then either $L = \text{gl}(V)$, $L = \text{sl}(V)$, $\dim V = 2r \geq 4$ and $L = \text{sp}(V)$, or $\dim V = 2r \geq 4$ and $L = \text{csp}(V)$.

Theorem 2, which is important in the study of primitive pseudo-groups and infinite Lie algebras of Cartan type, has been proved by several authors ([1], [2], [3], [5], [9]). These proofs have involved considerable use of the classification and representation theory of
semisimple Lie algebras over \( \mathbb{C} \) and hence cannot be generalized to fields of prime characteristic. In the proof presented here we use more elementary techniques which are valid over algebraically closed fields of characteristic \( \neq 2, 3, 5 \).

We will consider the following three conditions on a Lie algebra \( L \) of linear transformations of a vector space \( V \):

**Condition A.** There exists \( A \in L \) with rank \( A = 1 \) and \( A^2 \neq 0 \).

**Condition B.** Either \( \dim V = 2 \) or there exist \( A, B \in L \) with rank \( A = \ker A = \ker B \), and \( VA \neq VB \).

**Condition C.** There exists \( A \in L \) with \( \text{ad} A \neq 0 = (\text{ad} A)^2 \).

By Condition \( \sim X \) we will mean the negation of Condition \( X \). Theorem 1 is clearly a consequence of the following two lemmas.

**Lemma 1.** If \( \Phi \) is an algebraically closed field of arbitrary characteristic, \( V \) is a finite dimensional vector space over \( \Phi \), and \( L \subseteq \mathfrak{gl}(V) \) is a Lie algebra of infinite type then \( L \) contains a rank one transformation.

**Lemma 2.** Let \( V \) be a finite dimensional vector space over a field \( \Phi \) of characteristic \( \neq 2, 3, 5 \). Let \( L \) be an irreducible Lie algebra of linear transformations of \( V \). Assume that \( L \) contains a rank one transformation. Then:

(i) If Condition A holds \( L = \mathfrak{gl}(V) \).

(ii) If Conditions \( \sim A, B, \) and \( \sim C \) hold then \( L = \mathfrak{sl}(V) \).

(iii) If Conditions \( \sim A, \sim B, \) and \( \sim C \) hold then \( \dim V = 2r \geq 4 \) and \( L = \mathfrak{sp}(V) \) or \( L = \mathfrak{csp}(V) \).

It is shown in [2] that if \( \Phi = \mathbb{C} \) Lemma 1 is a consequence of Hilbert's Nullstellensatz. The proof given there is in fact independent of the assumption \( \Phi = \mathbb{C} \) and could be used to prove our Lemma 1. We will give a somewhat more elementary proof here. Part (i) of Lemma 2 is proved in [6] for the case \( \Phi = \mathbb{C} \).

In the proof of Lemma 1 we will need:

**Lemma 3.** Let \( \Phi \) be an algebraically closed field of arbitrary characteristic. Let \( V \) and \( W \) be finite dimensional vector spaces over \( \Phi \) with \( 2 \leq \dim V \leq \dim W \). Let \( T \) be a subspace of \( \text{Hom}(V, W) \) such that \( \dim T \leq \dim W \). Then there exist \( \phi \in T \) and \( v \in V \) such that \( \phi \neq 0 \), \( v \neq 0 \), and \( \phi v = 0 \).

**Proof.** Suppose \( T \subseteq \text{Hom}(V, W) \) is a counterexample to the lemma such that \( \dim W \) is minimal among all counterexamples. As \( T \) is a counterexample we have, for any nonzero \( v \in V \), \( \dim W \geq \dim vT = \dim T \geq \dim W \). Thus for any \( w \in W \) there is a unique \( \phi \in T \) such that \( \phi v = w \). Then if \( v_1, v_2 \) are linearly independent elements of \( V \) we can find a basis \( \{ \psi_1, \cdots, \psi_n \} \) of \( T \) such that
\( \nu_2 \psi_i = \nu_1 \psi_{i+1} \) for all \( 1 \leq i \leq n - 1 \), and

\[
\nu_2 \psi_n = \sum_{i=1}^{n} \nu_1 \psi_i a_i \quad \text{where the } a_i \in \Phi.
\]

(For if linearly independent elements \( \psi_1, \ldots, \psi_j \) satisfying (1) have been found for some \( 1 \leq j \leq n-1 \) then the minimality of \( \dim W \) implies that \( \nu_2 \psi_j \subseteq (\nu_1 \psi_1, \ldots, \nu_1 \psi_j) \). Thus there exists a unique \( \psi_{j+1} \in T \) such that \( \psi_1, \ldots, \psi_{j+1} \) are linearly independent and satisfy (1).

Proceeding by induction on \( j \) gives the result.) Now there exists \( \lambda \in \Phi \) such that \( \lambda^n - \sum_{i=0}^{n-1} \lambda^a_{i+1} = 0 \). Setting \( b_k = \lambda^{n-k} - \sum_{i=0}^{n-1} \lambda^a_{i+k+1} \) for \( 1 \leq k \leq n-1 \) we see that \( \psi = \psi_n + \sum_{i=0}^{n-1} \psi_i b_i \in T \) and \( (\nu_2 - \lambda \nu_n) \psi = 0 \). This contradicts the choice of \( T \) and proves the lemma.

Proof of Lemma 1. For \( \phi \in \mathcal{L}_i \) and \( j \leq i \) define

\[ \text{im}_j(\phi) = \langle \nu_j(\cdots (\nu_1 \phi) \cdots) \mid v_1, \ldots, v_j \in V \rangle. \]

Now if \( x \in \ker \phi \) and \( y \in V \) then \( 0 = y(x \phi) = x(y \phi) \) so we have \( \ker \phi \subseteq \ker(y \phi) \). Thus if \( \psi \in \text{im}_j(\phi) \) we have \( \ker \psi \supseteq \ker \phi \). Furthermore if \( d_i = \min \{ \text{rank } \phi \mid 0 \neq \phi \in \mathcal{L}_i \} \) we have \( d_0 \leq d_1 \leq \cdots \leq \dim V \). Hence there is some integer \( N \) such that \( d_N = d_{N+1} \) for all \( i \geq 0 \). Now if \( i \geq j \geq 0 \) and \( \phi \in \mathcal{L}_{N+i} \) satisfies \( \text{rank } \phi = d_N \) then for \( 0 \neq \psi \in \text{im}_j(\phi) \) we have \( \ker \psi \supseteq \ker \phi \) and \( \text{rank } \psi \leq d_N = \text{rank } \phi \). Thus \( \ker \psi = \ker \phi \). Thus \( \text{im}_j(\phi) \subseteq \text{Hom}(V/\ker \phi, \text{im}_{j+1}(\phi)) \) and \( 0 \neq \psi \in \text{im}_j(\phi) \) implies \( \text{rank } \psi = d_N = \dim(V/\ker \phi) \). Thus if \( d_N \geq 2 \), Lemma 3 shows that \( \dim \text{im}_j(\phi) \leq \dim \text{im}_{j+1}(\phi) - 1 \). Hence \( \dim \mathcal{L}_{N-1} \geq \dim \text{im}_{i+1}(\phi) \geq \dim \text{im}_i(\phi) + i \geq i \) for all \( i \geq 0 \). Since \( \mathcal{L}_{N-1} \) is finite dimensional this is impossible. Hence \( 1 = d_N = d_0 \), proving the lemma.

Proof of Lemma 2. Let \( n = \dim V \). If \( \{ x_1, \ldots, x_n \} \) and \( \{ y_1, \ldots, y_n \} \) are bases for \( V \) we define elements \( E_{ij} \in \mathfrak{gl}(V) \) for \( 1 \leq i, j \leq n \) by \( x_k E_{ij} = \delta_{ik} x_j \) and \( y_k F_{ij} = \delta_{kj} y_i \). If \( 2k \leq n \) we define \( \text{sp}(x_1, \ldots, x_{2k}) \) to be the Lie algebra of all \( A \in \mathfrak{gl}(V) \) such that \( VA \subseteq \langle x_1, \ldots, x_{2k} \rangle \), \( x_r A = 0 \) for all \( r > 2k \), and \( A \) is skew with respect to the skew-symmetric bilinear form defined by \( (x_{2i+1}, x_i) = \delta_{i+1} x_{i+1}, (x_{2i+2}, x_i) = -\delta_{i} x_{i+1} \), and \( (x_r, x_j) = 0 \) for all \( 0 \leq i \leq k-1, 1 \leq j \leq n, 2k < r \leq n \).

We will presently verify the following statements about an irreducible Lie algebra \( \mathcal{L} \) of linear transformations of \( V \):

(a) If \( 1 \leq k < n \) and \( E_{ij} \in \mathcal{L} \) for all \( 1 \leq i \leq k \) then there is a basis \( \{ y_1, \ldots, y_n \} \) of \( V \) such that \( F_{ij} \in \mathcal{L} \) for all \( 1 \leq i \leq k+1 \).

(b) If \( 1 \leq k < n \), \( E_{ij} \in \mathcal{L} \) for all \( 1 \leq i \leq n \), and \( F_{ij} \in \mathcal{L} \) for all \( 1 \leq i \leq k \) then there is a basis \( \{ y_1, \ldots, y_n \} \) of \( V \) such that \( F_{ij} \in \mathcal{L} \) for all \( 1 \leq i \leq n \) and \( F_{ij} \in \mathcal{L} \) for all \( 1 \leq i \leq k+1 \).
(c) If $2 \leq k \leq m \leq n$, $E_{it} \in L$ for all $2 \leq i \leq m$, $E_{it} \in L$ for all $2 \leq i < k$, and $L$ satisfies Condition $\sim C$ then there is a basis $\{y_1, \ldots, y_n\}$ of $V$ such that $F_{it} \in L$ for all $2 \leq i \leq m$ and $F_{it} \in L$ for all $2 \leq i \leq k$.

(d) If $3 \leq k < n$ and $E_{it}$, $E_{it} \in L$ for all $2 \leq i \leq k$ then there is a basis $\{y_1, \ldots, y_n\}$ of $V$ such that $F_{it} \in L$ for all $2 \leq i \leq k + 1$ and $F_{it} \in L$ for all $2 \leq i \leq k$.

(e) If $\text{sp}(x_1, \ldots, x_{2k}) \subseteq L$ where $\dim V > 2k + 2$ and $L$ satisfies Conditions $\sim B$ and $\sim C$ then $\dim V \geq 2k + 2$ and there is a basis $\{y_1, \ldots, y_n\}$ of $V$ such that $\text{sp}(y_1, \ldots, y_{2k+2}) \subseteq L$.

(f) If $\text{sp}(V) \subseteq L$ and $L$ satisfies Condition $\sim B$ then $L = \text{sp}(V)$ or $L = \text{csp}(V)$.

Lemma 2 follows immediately from statements (a)-(f). For if Condition A holds we may choose a basis $\{x_1, \ldots, x_n\}$ for $V$ such that $E_{it} \in L$. Then by (a), (b), and induction on $k$ we see that $L = \text{gl}(V)$, proving (i). If Conditions $\sim A$, $\sim B$, and $\sim C$ hold we may choose a basis $\{x_1, \ldots, x_n\}$ of $V$ such that $E_{it} \in L$ and if $n \geq 3$ we may also arrange that $E_{it} \in L$. Then using (c), (d), and induction on $k$ we see that $L = \text{sl}(V)$, proving (ii). Finally if Conditions $\sim A$, $\sim B$, and $\sim C$ hold we may choose a basis $\{x_1, \ldots, x_n\}$ of $V$ such that $E_{it} \in L$. By (c) we may assume that $\text{sp}(x_1, x_2) \subseteq L$. Then by (e) and induction on $k$ we have $\dim V = 2r + 4$ and $\text{sp}(V) \subseteq L$. Then (f) proves (iii).

We now verify (a)-(f). Throughout we will let $A = \sum E_{i,j}a_{i,j}$ where the $a_{i,j} \in \Phi$.

(a): As $\langle x_1, \ldots, x_k \rangle$ is not an invariant subspace there exists $A \in L$ such that $a_{i,j} \neq 0$ for some $1 \leq i \leq k < j \leq n$. We may assume that $i = 1$ (replacing $A$ by $A(ad E_{1i})$ if $i \neq 1$) and that $a_{i,j} = 0$ whenever $r > 1$ or $j = 1$ (replacing $A$ by $(A(ad E_{1i}) - A)(ad E_{1i})/2$). Letting $\{y_1, \ldots, y_n\}$ be any basis for $V$ satisfying $y_i = x_i$ for $1 \leq i \leq k$, $y_{k+1} = x_1A$, and $y_j \in \langle x_{k+1}, \ldots, x_n \rangle$ for $j > k + 1$ gives the result.

The proof of (b) is similar to that of (a).

(c): First assume that $ad E_{1k} = 0$. Then for every $A \in L$ we have $0 = A(ad E_{1k}) = \sum E_{i,j}(a_{i,j}E_{1i} - E_{1i}a_{i,j})$. Hence $a_{k,i} = 0$ for all $i \neq k$ so $\langle x_k \rangle$ is an invariant subspace, contradicting the irreducibility of $L$. Hence $ad E_{1k} \neq 0$ so by Condition $\sim C$ we have $(ad E_{1k})^2 \neq 0$. Since $A(ad E_{1k})^2 = E_{1k}(-2a_{k1})$ we have $E_{1k} \in L(ad E_{1k})^2$. Since $E_{1k}^2 = 0$ we have $(ad E_{1k})^2 = 0$. Thus Lemma V.8.2 of [8] shows that there exists $A \in L$ such that $A(ad E_{1k})^2 = 2E_{1k}$ and $A(ad E_{1k})(ad A) = -2A$. Now

$$A(ad E_{1k})(ad A) = \sum_{i,j} E_{i,j} \left(2a_{k,j}a_{j1} - \delta_{11} \sum_r a_{kr}a_{rj} - \delta_{jk} \sum_r a_{ir}a_{r1}\right).$$
Thus we have $a_{kl} = -1$ and

\begin{equation}
(2) \quad a_{ij} = - a_{i1} a_{kj} + \left( \delta_{i1} \sum_j a_{kr} a_{rj} + \delta_{jk} \sum_r a_{ir} a_{rl} \right) / 2.
\end{equation}

Setting $i = j = 1$ gives $0 = \sum_r a_{kr} a_{rl}$. From this, using (2) to substitute for $a_{rj}$ in $\sum_r a_{kr} a_{rj}$, we conclude that $\sum_r a_{kr} a_{rj} = 0$ for all $1 \leq j \leq n$ and similarly that $\sum_r a_{ir} a_{rl} = 0$ for all $1 \leq i \leq n$. Thus $a_{i1} = - a_{i1} a_{kj}$ and $\sum_r a_{ir} a_{rl} = 0$ for $1 \leq i, j \leq n$. Now set $y_1 = x_b A$, $y_b = x_b$, and $y_j = x_j + a_{j-1} x_j$ for $j \neq 1, k$. Then $F_{k1} = A \in L$, $F_{1b} = - E_{1b} \in L$, and $F_{ij} = - E_{ib} a_{ij} - E_{ti} \in L$ for $2 \leq i \leq m$, $i \neq k$. Finally, for $2 \leq i < k$ we have $F_{1i} = - E_{i1}(\text{ad} F_{k1})(\text{ad} F_{1k}) - (F_{11} - F_{kk}) a_{ki} \in L$ as required.

(d): As in (a) we may find a basis $\{y_1, \ldots, y_n\}$ of $V$ such that $F_{1i}, F_{1b} \in L$ for $1 \leq i \leq k$ and $A = \sum_i F_{ij} b_{ij} \in L$ where $b_{ij} = \delta_{i,j+1}$. Then $F_{1,k+1} = A(\text{ad} F_{k1})(\text{ad} F_{1k})(\text{ad} F_{1b}) + (F_{11} - F_{kk}) b_{21} \in L$ as required.

(e): It is well known (for example [8, p. 67]) where we take $x_{2r+1} = x_{r+1}$ and $x_{2r+2} = x_{r+1}$ for $0 \leq i \leq r - 1$ that sp($x_1, \ldots, x_{2r}$) is spanned by the following elements and their transposes: $E_{2r+1,2r+1}$, $E_{2r+1,2r+2}$, $E_{2r+1,2r+3}$, $E_{2r+1,2r+4}$, $E_{2r+1,2r+5}$, and $E_{2r+1,2r+6}$ for $0 \leq i \neq j \leq r - 1$. Thus it is easily checked that sp($x_1, \ldots, x_{2r}$) is generated by sp($x_1, \ldots, x_{2k}$), $E_{1,2k+1} - E_{2k+1,2k+1}$, and $E_{2k+1,2k+2}$.

We first show that for some basis $\{y_1, \ldots, y_n\}$ of $V$ we have sp($y_1, \ldots, y_{2k}$) $\subseteq L$ and $A = F_{1,2k+1} + \sum_{i>2k} F_{ij} b_{ij} \in L$ where the $b_{ij}$ are not an invariant subspace there exists $A \in L$ such that $a_{ij} \neq 0$ for some $1 \leq i \leq 2k < j \leq n$. We may assume that $i = 1$ (replacing $A$ by $A(\text{ad} E_{11})$ if $i = 2$, by $A(\text{ad}(E_{1,2r+1} - E_{2r+1,2}))$ if $i = 2r+1$, and by $A(\text{ad}(E_{1,2r+2} + E_{2r+1,2}))$ if $i = 2r+2$), that $a_{i1} = 0$ unless $r = 1$, $s > 2$ or $r > 2$, $s = 2$ (replacing $A$ by $A(\text{ad} E_{12})(\text{ad} E_{22}) - (E_{11} - E_{22}) (a_{11} - a_{22}) - E_{12} (2a_{12})$), and that $a_{is} = a_{si} = 0$ for $s \leq 2k$ (replacing $A$ by

\[ A - \sum_{i=1}^{k-1} A((\text{ad} E_{2i+1,2i+1}))(\text{ad} E_{2i+1,2i+2}) + (\text{ad} E_{2i+1,2i+2} \text{ad} E_{2i+1,2i+1})). \]

Then letting $\{y_1, \ldots, y_n\}$ be any basis for $V$ satisfying $x_1 = y_1$ for $1 \leq i \leq 2k$, $y_{2k+1} = x_1 A$, and $y_j \in (x_{2k+1}, \ldots, x_n)$ for $j > 2k + 1$ gives the result.

Now assume that sp($x_1, \ldots, x_{2k}$) $\subseteq L$ and that $A = E_{1,2k+1} + \sum_{i>2k} E_{ij} a_{ij} \in L$. 

We will show that for some basis \( \{ y_1, \ldots, y_n \} \) we have \( \text{sp}(y_1, \ldots, y_{2k}) \subseteq L \) and \( F_{1, 2k+1} - F_{2k+2, 2} \in L \). If \( a = a_{2k+1, 2} \neq 0 \) setting \( y_i = x_i \) for \( 1 \leq i \leq 2k + 1 \) and \( y_i = x_j - a_{2k+1, 2} x_{2k+1} \) for \( j > 2k + 1 \) we see that \( \text{sp}(y_1, \ldots, y_{2k}) \subseteq L \) and \( A = F_{1, 2k+1} + F_{2k+1, 2}a \). Then \( F_{2i}(ad \ A)^2 = (F_{1, 2k+1} - F_{2k+1, 2}a)3a \in L \) so \( F_{1, 2k+1} \in L \). This contradicts Condition \( \sim B \) so we must have \( a = 0 \). Also by Condition \( \sim B \) some \( a_{ij} \neq 0 \). Hence we can choose \( y_{2k+2} \in \langle x_{2k+2}, \ldots, x_n \rangle \) so that \( y_{2k+2} A = -x_2 \). Then setting \( y_i = x_i \) for \( 1 \leq i \leq 2k + 1 \) and choosing \( y_j \in \langle x_{2k+2}, \ldots, x_n \rangle \cap \ker A \) for \( j > 2k + 2 \) so that \( \{ y_1, \ldots, y_n \} \) is a basis for \( V \) we have the result.

Now assume \( \text{sp}(x_1, \ldots, x_{2k}) \subseteq L \) and \( E_{1, 2k+1} - E_{2k+2, 2} \in L \). We will show that for some basis \( \{ y_1, \ldots, y_n \} \) of \( V \) we have \( \text{sp}(y_1, \ldots, y_{2k}) \subseteq L \) and \( F_{1, 2k+1} - F_{2k+2, 2}, F_{2k+1, 2k+2} \in L \), thus proving (e). We have \( E_{2k+2, 2k+1} = E_{2i}(ad(E_{1, 2k+1} - E_{2k+2, 2}))^2/2 \in L \). Then as in (c) we have \( B = \sum b_{ij}E_{ij} \subseteq L \) where \( b_{ij} \in \Phi \) and satisfy \( b_{i+1, j+1} = -1, b_{ij} = -x_{2k+1}B, y_{2k+1} = x_{2k+1} \), and \( y_i = x_i + b_{i+1, 2k+1} x_{2k+1} \) for \( j \neq 2k+1, 2k+2 \) we have \( F_{2k+1, 2k+2} = -B \in L, F_{2k+1, 2k+1} = E_{2k+2, 2k+1} E_{1, 2k+1} E_{2k+1, 2k+2} \subseteq L \), and \( F_{1, 2k+1} - F_{2k+2, 2} = E_{1, 2k+1} - E_{2k+2, 2} - E_{2k+2, 2k+1}(b_{2k+2, 2k+2} - b_{1, 2k+1}) \in L \). Also, for \( 1 \leq i, j \leq 2k \), we have

\[
F_{ij} = E_{ij} - E_{ij}(ad F_{2i+1, 2j+1})(ad F_{2i+2, 2j+1}) - F_{2i+2, 2j+1}b_{ij}.
\]

Hence \( \text{sp}(y_1, \ldots, y_{2k}) \subseteq L \) and \( F_{1, 2k+1} - F_{2k+2, 2}, F_{2k+1, 2k+2} \in L \) as required.

(f): If \( \text{dim } V = 2k \) and \( \text{sp}(V) \subseteq L \) then for \( 0 \leq i \neq j \leq k - 1 \) if \( A \in L \) then

\[
E_{2i+1, 2j+1}a_{2i+1, 2j+1} + E_{2j+2, 2i+2}a_{2j+2, 2i+2} = A(ad E_{2i+1, 2j+1})(ad E_{2j+2, 2i+2})(ad E_{2i+2, 2j+1}) \subseteq L.
\]

Then by Condition \( \sim B \) we must have \( a_{2i+1, 2j+1} + a_{2j+2, 2i+2} = 0 \). Similarly we see that \( a_{2i+1, 2j+2} = a_{2j+2, 2i+1} \) and \( a_{2i+2, 2j+1} = a_{2j+1, 2i+2} \). Hence \( A = D + S \) where \( S \in \text{sp}(V) \) and \( D = \text{diag} \{ d_1, \ldots, d_{2k} \} \in L \).

Now

\[
D(ad(E_{1, 2i+1} - E_{2i+2, 2})) = E_{1, 2i+1}(d_1 - d_{2i+1}) - E_{2i+2, 2}(d_{2i+2} - d_2) \in L.
\]

Thus, again by Condition \( \sim B \), we have \( d_1 - d_{2i+1} = d_{2i+2} - d_2 \) for all \( 0 \leq i \leq k - 1 \). Thus \( 2D = I(d_1 + d_2) + E \) where \( E \in \text{sp}(V) \). Thus \( L = \text{sp}(V) \) or \( L = \text{csp}(V) \).

References


