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ON THE MEAN-VALUE PROPERTY OF HARMONIC FUNCTIONS

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ABSTRACT. In this note we show that if the areal mean-value theorem holds for a plane domain (subject to a mild regularity condition) for all integrable harmonic functions, then the domain must be a disk. It is also shown that if a plane domain with finite area has at least two boundary components which are continua then the mean-value property cannot hold for the class of all integrable harmonic functions with single-valued harmonic conjugates.

1. In 1962 Epstein[1] [2] proved the following theorem: "Let D be a simply connected domain of finite area and t a point of D such that, for every function u harmonic in D and integrable over D, the mean-value of u over the area of D equals $u(t)$. Then D is a disk and t its center." In a later paper Epstein and Schiffer extended the above result to domains in Euclidean space E^n replacing the simple connectivity hypothesis of the earlier paper by the assumption that the complement of D possess a nonempty interior. Nevertheless, the following theorem strongly suggests that, for plane regions at least, the simple connectedness of D is a necessary condition for the mean-value property to hold.

THEOREM 1. *Let D be a plane domain of finite area having at least two boundary components γ_1 and γ_2 which are continua. Denote by $\mathcal{H}$ the class of functions u harmonic in D and integrable over D; and by $\mathfrak{F}\subset\mathcal{H}$ the subclass consisting of functions possessing single-valued harmonic conjugates in D. Then $\mathfrak{F}$ (and a fortiori $\mathcal{H}$) does not satisfy the mean-value property at any point $t\in D$; that is, it is not the case that there exists a $t\in D$ such that, for all $u\in\mathfrak{F}$,*

$$u(t) = \frac{1}{A}\int\int_D u(z)\,dxdy,$$

where A denotes the area of D.

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2. Before proving Theorem 1 we will first establish some preliminary results. Given a point t in a plane region W we denote by p_0 and p_1 the principal functions (see, for example, [5]) with respect to the normal operators L_0 and L_1 and the given singularity t. Here p_1 is defined with respect to the canonical partition Q. The functions P_0 and P_1 are defined by

$$P_\nu(z; t) = p_\nu(z; t) + ip_\nu^*(z; t), \qquad \nu = 0, 1,$$

with

$$\lim_{\tau\to 0} (P_\nu(\tau; t) - 1/\tau) = 0,$$

where τ is a parameter about t and p_ν^* is the conjugate harmonic function of p.

Let $\tilde{K}(z; t)$ be the kernel function for the class $l_2(D)$ of square integrable analytic functions f in a domain D which possess single-valued integrals in D. Then we have

LEMMA 1. $\tilde{K}(z; t) = (1/2\pi)\ (P_0'(z; t) - P_1'(z; t))$.

A proof of this relation may be found in [6]. We also have

LEMMA 2. *Let W be a plane domain of connectivity $n \geqq 2$, possessing at least two boundary components γ_1, γ_2, which are disjoint analytic Jordan curves. Then the function $\frac{1}{2}(P_0 - P_1)$ is not univalent in W.*

PROOF. The function $\frac{1}{2}(P_0 - P_1)$ is analytic in $W \cup \gamma_1 \cup \gamma_2$. Along γ_i, $i = 1, 2$,

$$d(P_0 + P_1) - \overline{d(P_0 - P_1)} = 2(dp_1 + i * dp_0) = 0.$$

Therefore, on γ_i, $i = 1, 2$,

$$\tfrac{1}{2}(P_0 - P_1) = \tfrac{1}{2}\overline{(P_0 + P_1)} + \text{const},$$

where the constant depends on γ_i. Denote by γ_i' the image of γ_i under $\frac{1}{2}(P_0 - P_1)$. Due to the mapping properties of $\frac{1}{2}(P_0 + P_1)$, we see that if $\frac{1}{2}(P_0 - P_1)$ is univalent and if the connectivity n is $\geqq 2$, then either γ_1' or γ_2' is not the outer boundary, the region encircled by it being disjoint from the image region. Since this is a contradiction, we conclude that $\frac{1}{2}(P_0 - P_1)$ is not univalent.

3. **Proof of Theorem 1.** Assume to the contrary that $\mathfrak{F}$ satisfies the mean-value property at a point $t \in D$. By the Schwarz inequality we observe that for each $f(z) \in l_2(D)$, the real and imaginary parts of $f(z)$ are integrable over D, and so

$$f(t) = \iint_D f(z) \cdot A^{-1}\, dxdy.$$

On the other hand the equality $f(t) = \iint_D f(z)\overline{\tilde{K}(z;\, t)}dxdy$ holds for each $f \in l_2(D)$. Since $\tilde{K}(z; t)$ is uniquely determined by its reproducing property we conclude that

$$\tilde{K}(z; t) = A^{-1}.$$

This last relation together with Lemma 1 imply that $\frac{1}{2}(P_0 - P_1)$ is a linear function of z and hence univalent. If γ_1 and γ_2 are not both disjoint analytic Jordan curves to begin with we can by repeated applications of the Riemann mapping theorem map D conformally onto a domain D with finite area such that γ_1 and γ_2 are mapped onto disjoint analytic Jordan curves. We note that if ϕ is the one-to-one conformal map of D onto $\hat{D}$ then $\hat{P}_0 = P_0 \circ \phi^{-1}$ and $\hat{P}_1 = P_1 \circ \phi^{-1}$ are the corresponding principal functions for $\hat{D}$. Since $\frac{1}{2}(P_0 - P_1)$ is univalent on D it follows that $\frac{1}{2}(\hat{P}_0 - \hat{P}_1)$ is univalent on $\hat{D}$. But by Lemma 2 this is a contradiction. Hence $\mathfrak{F}$ does not satisfy the mean-value property at the point t as claimed.

4. We now turn to the theorem of Epstein and Schiffer [**3**] which is as follows:

THEOREM 2. *Let D be any domain in Euclidean space E^n, possessing finite measure, and let the complement of D possess nonempty interior. Suppose that there exits a point t in D such that, for every function u harmonic in D and integrable over D, the mean-value of u over D equals $u(t)$. Then D is a sphere with center at t.*

Epstein and Schiffer remark at the end of their paper that due to the assumption made about the complement of D, Theorem 2 leaves open the possibility that there exists a domain D which has finite area, is dense in E^n, and for which the mean-value property stated in their theorem holds. We now show that for plane regions we can give a sharper formulation of Theorem 2 in that the assumption that the complement of D possess nonempty interior may be replaced by a weaker assumption. Because of this result we are able partially to answer the question posed by Epstein and Schiffer above. Note that the class of functions considered in Theorem 2 corresponds to the class $\mathfrak{H}$ considered in Theorem 1 when $n = 2$.

A plane point set E which is compact is said to belong to the class N_B if the unbounded component of the complement of E belongs to the class O_{AB} of Riemann surfaces which possess no nonconstant bounded analytic functions (cf. Ahlfors-Beurling [**1**]). Sets in the class N_B are totally disconnected. We now state

THEOREM 3. *Let D be a plane domain with finite area having at least one boundary component γ which is a continuum. If $\mathcal{H}$ satisfies the mean-value property at a point $t \in D$ then D is a disk.*

PROOF. Since the mean-value property holds at the point t for $\mathcal{H}$ it holds for the subclass $\mathfrak{F}$ of functions possessing single-valued harmonic conjugates in D. As in the proof of Theorem 1 we see that this implies that $\frac{1}{2}(P_0 - P_1)$ is univalent. Because of Theorem 2 we may assume without loss of generality that D is dense in the entire plane. When this is so there are two possibilities to consider:

Case 1. Suppose that the complement of D with respect to the extended plane is the union $\gamma \cup A$, where A is the union of an at most countable number of compact sets of class N_B. Since a set of type N_B has 2-dimensional Hausdorff measure zero (cf. Sario-Oikawa [**5**]), and hence zero area, A has zero area. Also D has finite area by assumption and so γ must have infinite 2-dimensional Hausdorff measure. But this is impossible. Hence Case 1 does not in fact occur.

Case 2. Suppose now that the complement of D with respect to the extended plane is the union $\gamma \cup B$, where B is not an at most countable union of sets of type N_B. By the Riemann mapping theorem the extended plane less γ can be mapped conformally onto a disk with γ being mapped onto the bounding circle. The image of B under this mapping is a set $\hat{B}$ which is again not an at most countable union of sets of type N_B. But by Theorems 1, 2 of Sakai [**4**] this implies that $\frac{1}{2}(P_0 - P_1)$ is not univalent. It follows that Case 2 also does not occur, and this concludes the proof.

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