PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 29, No. 2, July 1971

PERFECT MATRIX METHODS

D. J. FLEMING AND P. G. JESSUP

ABSTRACT. Let $e_i = (\delta_{ij})_{j=1}^{\infty}$, $\Delta = (e_i)_{i=1}^{\infty}$ and let A be an infinite matrix which maps E into E where E is an FK-space with Schauder basis Δ. Necessary and sufficient conditions in terms of the matrix A are obtained for E to be dense in the summability space $E_A = \{x \mid Ax \in E\}$ and conditions are obtained to guarantee that E_A has Schauder basis Δ. Finally it is shown that if weak and strong sequential convergence coincide in E then in E_A the series $\sum_k x_k e_k$ converges to x strongly if and only if it converges to x weakly.

1. **Introduction.** If x is a sequence of scalars and $A = (a_{nk})$ is an infinite matrix then by Ax, the A-transform of x, we mean the sequence y, where $y_n = (Ax)_n = \sum_k a_{nk} x_k$ provided each of these sums converge. If E is any FK-space then E_A denotes the collection of all sequences x such that $Ax \in E$. The space E_A inherits a topology which makes it into an FK-space [5, p. 226]. A matrix A with the property that $Ax \in E$ whenever $x \in E$ will be called an E-E method. If A is an E-E method then $E \subseteq E_A$; if in addition $\overline{E} = E_A$ then A is called perfect. Let ϕ denote the space of all finitely nonzero sequences, l the space of absolutely summable sequences (with $\|x\| = \sum_k |x_k|$) and $\Delta = (e_i)_{i=1}^{\infty}$, where e_i is the sequence $(\delta_{ij})_{j=1}^{\infty}$.

In [3] it is shown that a reversible l-l method is perfect if and only if the matrix A has no nonzero left annihilators in m, the space of bounded sequences. In [2] conditions are obtained for a general l-l method to be perfect. It is also shown in [2] that the series $\sum_k x_k e_k$ converges strongly to $x \in l_A$ if and only if it converges weakly to x. The purpose of this note is to show that many of the results obtained in [2] and [3] for the summability field of an l-l method carry over to the summability field of an E-E method when E is an FK-space with basis Δ. In particular we show (Theorem 9) that if weak and strong sequential convergence coincide in E then for $x \in E_A$ the series $\sum_k x_k e_k$ converges to x strongly if and only if it converges weakly and (Theorem 2) that a reversible E-E method A is perfect if and only if A has no nonzero left annihilators in the sequence space representation of its dual. We will assume throughout this note that E

Received by the editors July 1, 1970.

AMS 1969 *subject classifications*. Primary 4031, 4046.

Key words and phrases. Perfect matrix method, associative matrix method, property P, type M, type M^*.

is an FK-space with basis Δ and so in particular every such space contains ϕ.

2. **Notation and terminology.** An E-E method is said to be reversible if the equation $y=Ax$ has a unique solution x for each $y\in E$. If A is a reversible E-E method then E_A is topologically isomorphic to E under the map A [**5**, Corollary 5, p. 204, Corollary 1, p. 199]. If the E-E method A is reversible then every $f\in E_A^*$ can be written in the form $g\circ A$ for $g\in E^*$, where * denotes the space of continuous linear functionals.

If x and y are sequences then (x, y) will denote the sum $\sum_k x_k y_k$ and xA denotes the sequence $(\sum_n x_n a_{nk})_{k=1}^{\infty}$. For E an FK-space let $E^\delta=\{t_f \mid f\in E^*\}$, where $t_f=(f(e_n))_{n=1}^{\infty}$. Let bs denote the set of sequences with finite norm $\|x\|=\sup_n\left|\sum_{j=1}^n x_j\right|$, cs the set of sequences x for which $\sum_k x_k$ converges with the norm inherited from bs, bv the space of sequences of bounded variation with $\|x\|=|x_1|+\sum_k|x_k-x_{k+1}|$, c_0 the sequences which converge to zero with the sup norm and $bv_0=bv\cap c_0$ with the norm of bv. Each of the above is a BK-space. Finally we let ω denote the FK-space of all scalar sequences with the product topology.

3. **Principal results.** Motivated by the notions of type M, type M^* (see, for example, [**1**, p. 90], [**4**, p. 184] and [**2**, p. 358]) and the fact that $l^\delta=m$ and $c^\delta=l$ we make the following definition.

DEFINITION 1. An E-E method A is said to be of type E^δ if whenever $tA=0$ for $t\in E^\delta$ then $t=0$.

THEOREM 2. *Let A be a reversible E-E method; then A is perfect if and only if A is of type E^δ.*

PROOF. ($\Leftarrow$) It suffices to show that Δ is a fundamental set in E_A. Let $f\in E_A^*$ and suppose that $f(e_k)=0$ for each k. Since $f\in E_A^*$ there exists a $g\in E^*$ with $f=g\circ A$. Thus $0=f(e_k)=g[Ae_k]=g[(a_{1k}, a_{2k}, \cdots)]$ for each k. For $g\in E^*$ and $x\in E$, $g(x)=\sum_n g(e_n)x_n$ and hence $\sum_n g(e_n)a_{nk}=0$ for each k. Since A is of type E^δ it follows that $g(e_n)=0$ for each n and hence $g\equiv 0$. Thus for $x\in E_A$, $f(x)=g[Ax]=0$ and so Δ is a fundamental set in E_A.

($\Rightarrow$) Assume now that $\bar{E}=E_A$ and that $t_fA=0$ for some $f\in E^*$. Let F_a denote the E_A topology and let $A|E$ denote A considered as a linear operator from E into E. Since $A:E_A\to E$ is continuous and $f\in E^*$ it follows that $f\circ A|E\in(E, F_a)^*$. Furthermore Δ is a basis for (E, F_a) since the F_a topology is weaker than the topology of E [**5**, p. 203]. Now $f[Ae_k]=f[\sum_n a_{nk}e_n]=\sum_n a_{nk}f(e_n)=(t_fA)_k$.

Therefore $\phi \subseteq (f \circ A | E)^{\perp}$ but $(f \circ A | E)^{\perp}$ is F_a-closed in E and ϕ is F_a-fundamental in E, hence $f \circ A | E \equiv 0$. The zero functional and $f \circ A$ are both continuous extensions of $f \circ A | E$ to all of E_A. Since $\bar{E} = E_A$ it follows that $f \circ A \equiv 0$ and hence by the reversibility of A, $f \equiv 0$. Thus $t_f = 0$ and A is of type E^f.

Since $l^\delta = m$ we obtain as a corollary the following theorem of Brown and Cowling [**3**, Theorem 2].

COROLLARY 3. *A reversible l-l method is perfect if and only if it is of type M^*.*

Similarly for reversible E-E methods, where E is one of the familiar sequence spaces cs, c_0 or bv_0, we have that perfectness is equivalent to type bv, type l, and type bs respectively.

DEFINITION 4. If A is an E-E method and $t \in E^\delta$ we say that t has property P if $(tA, x) = \sum_k \sum_n t_n a_{nk} x_k$ converges for each $x \in E_A$. The set of all $t \in E^\delta$ with property P is denoted by Q. The method is called associative if $Q = E^\delta$ and $f[Ax] = (t_f A, x)$ for each $f \in E^*$ and each $x \in E_A$ (cf. [**2**, p. 282]).

LEMMA 5. *Let A be an E-E method and let $t \in Q$ then $(tA, \cdot)$ defines a continuous linear functional on E_A.*

PROOF. Let $g_j = \sum_{k=1}^{j} (\sum_n t_n a_{nk}) E_k$ and $g(x) = (tA, x)$, where E_k is the kth coordinate functional. Since E_A is an FK-space $g_j \in E_A^*$ for each j and since $t \in Q$, $g_j \to g$ pointwise on E_A. The continuity of g follows from [**5**, p. 200].

THEOREM 6. *Let A be an E-E method. Then A is perfect if and only if $f[Ax] = (t_f A, x)$ for each $x \in E_A$ and each $t_f \in Q$ (cf. [**3**, Theorem 1] and [**2**, Theorem A]).*

PROOF. ($\Rightarrow$) Let $t_f \in Q$ and let $g(x) = (t_f A, x)$ for $x \in E_A$; then $f[Ae_j] = \sum_i a_{ij} f(e_j) = (t_f A, e_j)$ and so $g = f \circ A$ on the fundamental set Δ. Since g and $f \circ A$ are continuous on E_A it follows that $g \equiv f \circ A$.

($\Leftarrow$) Let $f \in E_A^*$ be such that $f(e_k) = 0$ for each k. By [**5**, p. 230] there exists $F \in \omega_A^*$ and $G \in E^*$ such that $f(x) = F(x) + G[Ax]$ for each $x \in E_A$. Therefore $0 = f(e_k) = F(e_k) + G[\sum_n a_{nk} e_n] = F(e_k) + \sum_n a_{nk} G(e_n)$. Since Δ is a basis for ω_A [**5**, p. 230] we have in particular that $F(x) = \sum_k F(e_k) x_k$ for each $x \in E_A$. Combining these results we have that

$$\sum_k F(e_k) x_k = - \sum_k \left(\sum_n G(e_n) a_{nk} \right) x_k$$

for each $x \in E_A$. Thus

$$\begin{aligned} f(x) &= F(x) + G[Ax] \\ &= \sum_k F(e_k)x_k + \sum_n G(e_n) \sum_k a_{nk}x_k \\ &= \sum_n G(e_n) \sum_k a_{nk}x_k - \sum_k \left(\sum_n G(e_n)a_{nk} \right) x_k \\ &= G[Ax] - (t_f A, x) = 0. \end{aligned}$$

Hence $f \equiv 0$ and so $\bar{E} = E_A$.

THEOREM 7. *Let A be an E-E method. Then A is associative if and only if E_A has basis Δ.*

PROOF. ($\Rightarrow$) Let $x \in E_A$ and $f \in E_A^*$. Choose $F \in \omega_A^*$, $G \in E^*$ such that $f = F + G \circ A$ and let $y_n = x - \sum_{k=1}^{n-1} x_k e_k$. Then

$$\begin{aligned} f(y_n) &= F(y_n) + G[Ay_n] = F(y_n) + (t_G A, y_n) \\ &= F(y_n) + \sum_{k=n}^{\infty} \left(\sum_{j=1}^{\infty} G(e_j)a_{jk} \right) x_k. \end{aligned}$$

The first term limits to 0 on n since Δ is a basis for ω_A and the second limits to 0 since the double series converges. Thus Δ is a weak basis and hence a basis for E_A.

($\Leftarrow$) Let $x \in E_A$ and $f \in E^*$ then $f \circ A \in E_A^*$ and so

$$f[Ax] = \sum_k x_k f[Ae_k] = \sum_k x_k \sum_n a_{nk} f(e_n) = (t_f A, x).$$

We shall say that $x \in E_A$ is perfect if $f(Ax) = (t_f A, x)$ for each $t_f \in Q$ and that x is associative if $Q = E^\delta$ and $f(Ax) = (t_f A, x)$ for all $t_f \in Q$.

THEOREM 8. *Let A be an E-E method and let $x \in E_A$; then*

(i) *$\sum_k x_k e_k$ converges to x weakly if and only if x is associative,*

(ii) *x is in the closure of ϕ in E_A if and only if x is perfect.*

PROOF. (i) ($\Rightarrow$) Let $t_f \in E^\delta$ and let $F = f \circ A$; then $F \in E_A^*$ and $F(x) = \sum_k x_k F(e_k) = \sum_k x_k f(Ae_k) = \sum_k x_k \sum_n a_{nk} f(e_n) = (t_f A, x)$.

($\Leftarrow$) Let $g \in E_A^*$; say $g = F + G \circ A$ for $F \in \omega_A^*$ and $G \in E^*$; then $g(e_k) = F(e_k) + \sum_n G(e_n) a_{nk}$. Thus

$$\begin{aligned} g(x) &= F(x) + G[Ax] = \sum_k x_k F(e_k) + G[Ax] \\ &= \sum_k x_k \left(g(e_k) - \sum_n G(e_n) a_{nk} \right) + G[Ax] \\ &= \sum_k g(e_k) - (t_G A, x) + G[Ax] = \sum_k x_k g(e_k). \end{aligned}$$

(ii) ($\Rightarrow$) Let x be in the closure of ϕ in E_A and let $t_f \in Q$. Define $g: E_A \to k$ by $g(y) = f[Ay] - (t_f A, y)$; then $g \in E_A^*$ by Lemma 5 but $g(e_k) = 0$ for each k and so $g(x) = 0$. Therefore $f[Ax] = (t_f A, x)$.

($\Leftarrow$) Let $f \in E_A^*$ be such that $f | \phi \equiv 0$. Then, as in (i), $f(x) = \sum_k f(e_k) x_k + G[Ax] - (t_G A, x) = G[Ax] - (t_G A, x)$. Thus $t_G \in Q$ and so $f(x) = 0$.

For the following theorem we do not assume E has basis Δ.

THEOREM 9. *Let A be an E-E method and suppose that weak and strong sequential convergence coincide in E. Then for $x \in E_A$ the series $\sum_k x_k e_k$ converges to x if and only if it converges to x weakly.*

PROOF. Let $x \in E_A$ be such that $\sum_{k=1}^n x_k e_k \to x$ weakly and let $y_j = (0, \cdots, 0, x_j, x_{j+1}, \cdots)$. Let (r_n) be the determining seminorms for E; then the topology of E_A is given by the seminorms $(|E_n|)$, (p_n), (q_n), where $q_n = r_n \circ A$ and p_n is defined by

$$p_n(x) = \sup_m \left| \sum_{k=1}^m a_{nk} x_k \right| \quad [5, \text{p. } 226, \text{Theorem } 1].$$

Since $E_n \in E_A^*$ for each n it is clear that $|E_n(y_j)| \to_j 0$ for each n. Let $f \in E^*$ then $f \circ A \in E_A^*$ and hence $f \circ A(y_j) \to_j 0$. Thus $(A(y_j))$ converges to zero weakly and hence strongly in E and so $q_n(y_j) \to 0$ for each n. Finally fix n and let $\epsilon > 0$ be given. Choose N such that $j, m \geqq N$ implies

$$\left| \sum_{k=j}^m a_{nk} x_k \right| < \epsilon.$$

Thus

$$\sup_{m > j} \left| \sum_{k=j}^m a_{nk} x_k \right| \leqq \epsilon \qquad \text{for } j > N,$$

but $p_n(y_j) = \sup_m | \sum_{k=j}^m a_{nk} x_k |$ and hence $p_n(y_j) \to_j 0$ for each n. It follows that $y \to_j 0$ in E_A.

REMARKS. (i) It has been pointed out by G. Bennett that the proof of Lemma 3 on p. 285 of [2] makes incorrect use of Satz 3.4 of [6, p. 60]. Since weak and strong sequential convergence coincide in l Lemma 3 of [2] follows from Theorem 9 above.

(ii) If E is an FK-space with determining seminorms (r_n) and if A is a row finite E-E method then the seminorms $(|E_n|)$ and $(r_n \circ A)$ are sufficient to determine the topology of E_A. Thus if weak and strong sequential convergence coincide in E one can proceed as in the proof of Theorem 9 to show they coincide in E_A. This result has been observed by Bennett in [7].

References

1. S. Banach, *Théorie des opérations linéaires*, Monografie Mat., PWN, Warsaw, 1932; reprint, Chelsea, New York, 1955. MR **17**, 175.

2. H. I. Brown, *The summability field of a perfect l-l method of summation*, J. Analyse Math. **20** (1967), 281–287. MR **36** #1869.

3. H. I. Brown and V. F. Cowling, *On consistency of l-l methods of summation*, Michigan Math. J. **12** (1965), 357–362. MR **32** #1480.

4. M. S. Ramanujan, *On summability methods of type M*, J. London Math. Soc. **29** (1954), 184–189. MR **15**, 697.

5. A. Wilansky, *Functional analysis*, Blaisdell, New York, 1964. MR **30** #425.

6. K. Zeller, *Abschnittskonvergenz in FK-Räumen*, Math. Z. **55** (1951), 55–70. MR **13**, 934.

7. G. Bennett, *A representation theorem for summability domains*, Pacific J. Math. (to appear).

Clarkson College of Technology, Potsdam, New York 13676