ON RINGS SATISFYING \([ (a, b, c), d ] = 0 \)

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Abstract. A simple nonassociative ring, in which the associators commute with all elements, is under mild additional assumptions either associative or commutative. This result cannot be extended to prime rings since a construction of semiprime rings gives counterexamples.

Introduction. Let \( R \) be a nonassociative ring, in which the associators commute with all elements. If \( R \) is simple then we prove under mild additional assumptions that \( R \) is either associative or commutative. This result cannot be extended to prime rings. To prove this we present a construction of semiprime rings giving counterexamples. Our class of rings includes the standard algebras and accessible rings, which were studied by Albert and Kleinfeld and others.

Structure theory. Let \( R \) be a nonassociative ring. We denote the deviation from the commutative law by the commutator \([x, y] := xy - yx\) and the deviation from the associative law by the associator \((x, y, z) := (xy)z - x(yz)\). In the following we study rings \( R \) satisfying the relation

\[
\left[(a, b, c), d\right] = 0 \quad \text{for all } a, b, c, d \in R.
\]

The relation (*) holds in accessible and in standard rings \([4]\). If we put \( V := \{x \mid x \in R, [x, y] = 0 \text{ for all } y \in R\} \), then (*) is equivalent to saying that \( V \) contains all associators. In \([4], [5], [6], [9], [10], [12], [13]\) other generalizations of standard rings are considered. In most rings that were studied until now, \( V \) is a subalgebra of \( R \). Since we need this fact we state

**Lemma 1.** Let \( R \) be any ring. Then \( V = \{x \mid x \in R, [x, R] = 0\} \) is a subring of \( V \) if \( R \) satisfies one of the following conditions:

(a) \((x, x, y) + (y, x, x) - 2(x, y, x) = 0.\)

(b) \((x, y, z) + (y, z, x) + (z, x, y) = 0, x, y, z \in R.\)

(c) \((x, x, x) = 0 \text{ and } 2y = 0 \text{ only for } y = 0.\)

**Proof.** The conditions (a), (b), (c) can be rewritten by using commutators respectively as:

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\( (a') \quad [x^2, y] + [[[y, x], x] + [x, xy] + 3x[y, x] = 0. \)

\( (b') \quad [xy, z] + [yz, x] + [zx, y] = 0. \)

\( (c') \quad [x^2, x] = 0 \) and \( 2y = 0 \) implies \( y = 0. \)

From \( (b') \) or from the polarizations of \( (a') \) or \( (c') \) one easily sees that \( V \) is closed under multiplication, hence a subring of \( V \). \( V \) was also used in the proofs of [3]. An easy example shows that \( V \) is not always a subalgebra of \( R \). Clearly flexible rings, being defined by \( (x, y, x) = 0 \), satisfy \( (a) \). The rings of \((\gamma, \delta)\)-type [7] satisfy \( (b) \). In the following we will always assume that \( V \) is a subalgebra.

**Lemma 2.** If \( V \) is a subalgebra of \( R \) and \( R \) satisfies (*) then the set \( W := \{ v \in V, Rv \subseteq V \} \) is an ideal of \( R \) such that \((x, y, v) \in W \) and \((v, y, x) \in W \) for \( v \in V \) and all \( x, y \in R \).

**Proof.** From (*) we see that \( W \) is a (two-sided) ideal of \( R \). From the well-known identity

\[ a(ab, c, d) + (a, bc, d) - (a, b, cd), \]

which holds in any ring, we get

\[ z(x, y, v) = (zx, y, v) - (z, xy, v) + (z, x, yv) - (z, x, y)v \in V \]

since \( V \) is a subalgebra of \( R \) and (*) holds. Similarly we get \( z(v, y, x) = (v, y, x)z \in V \).

**Corollary 1.** The canonical homomorphism of \( R \) onto \( R/W \) maps \( V \) into the center of \( R/W \).

**Proof.** Let \( x, y \in R \) and \( v \in V \). We know already \([x, v] = 0, (x, y, v) \in W \) and \((v, y, x) \in W \) from Lemma 2. Therefore also

\[ (x, v, y) = (x, y, v) + (v, x, y) - [xy, v] + [x, v]y - x[y, v] \]

\[ = (x, y, v) + (v, x, y) \in W. \]

**Corollary 2.** If \( W = 0 \) then \( V \) equals the center of \( R \).

**Proof.** By Corollary 1, \( V \) is contained in the center while the other inclusion is trivial.

**Corollary 3.** \((x, y, z)^3 = (x, x, x) (y, y, y) (z, z, z) \) and \( 2(x, y, z)^3 \equiv 0 \) modulo \( W \).

**Proof.** Since \( V \) is mapped into the center of \( R/W \), we have modulo \( W \), that

\[ (x, y, z)^3 = (x, y, z)((x, y, z), y, z) = -(x, y, z)((x, x, y)z, y, z) \]

\[ = -(x(x, x, y), y, z)z, y, z) = (x, x, x)(y, y, z)(z, y, z). \]
Now  
\[(y, y, z)(z, y, z) = (z, y(y, y, z), z) = -(y, y, y)(z, z, z)\]
or  
\[(y, y, z)(z, y, z) = (z(y, y, z), y, z) = -(z, y(y, y)y, z) = (z, z(y, y, y), z) = (y, y, y)(z, z, z)\]
which shows \(2(y, y, y)(z, z, z) = 0\).

**Theorem 1.** Let \(R\) be a ring without nonzero ideals \(\neq R\) satisfying (*). If \(V\) is a subalgebra of \(R\) and \((x, x, x)\) is nilpotent for each \(x \in R\), then \(R\) is either associative or commutative.

**Proof.** The ideal \(W\) of Lemma 2 is contained in the commutative subalgebra \(V\) of \(R\). Since \(R\) has no nontrivial ideal either \(W = R\) or \(W = 0\). In the first case \(R\) is commutative. Let us therefore consider the case \(W = 0\). By Corollary 2 all associators are in the center of \(R\) and are nilpotent by Corollary 3. Therefore \(R(x, y, z)\) is a nilpotent ideal of \(R\). Hence \(R(x, y, z) = 0\). This implies \((x, y, z) \in W\) or \((x, y, z) = 0\).

**Remark.** In Lemma 1 we saw that the assumption that \(V\) is a subalgebra of \(R\) is a very mild restriction. From the proof of Corollary 3 we see \(2(x, x, x)^2 = 0\). Hence the assumption that \((x, x, x)\) is nilpotent is only necessary in case \(2y = 0\) for all \(y \in R\).

**Corollary.** Let \(R\) be a simple ring satisfying (*). Then \(R\) is commutative or associative if one of the following conditions is satisfied:

(a") \((x, x, y) + (y, x, x) - 2(x, y, x) = 0\) and \(2z = 0\) implies \(z = 0\).
(b") \((x, y, z) + (y, z, x) + (z, x, y) = 0\).
(c") \((x, x, x) = 0\) and \(2z = 0\) implies \(z = 0\).
(d") \((x, y, x) = 0\).

**Construction of semiprime rings.** In the preceding section we classified the simple rings satisfying (*). In similar cases, e.g. for rings with commutators in the nucleus, one can make a similar statement with simple weakened to prime or even semiprime \([5], [6], [9], [12], [13]\). This is not true for rings satisfying (*). We will give a construction of prime and semiprime rings to get counterexamples.

Let us recall that a ring is called prime if the product of any two (two-sided) nonzero ideals is nonzero. We call a ring semiprime, if the square of any nonzero ideal is nonzero. Clearly prime rings are semiprime.
Let $k$ be a field and $R$ a $k$-algebra with unit element $e$. We assume that $R$ has a subvectorspace $M \neq 0$ containing all commutators such that $R = ke + M$ is a direct sum. Let $ku$ be the $k$-vectorspace generated by some element $u \in R$. On the direct sum $R^u := ku + R$ we define a $k$-algebra structure by conserving the multiplication in $R$ and requiring $u^2 = u$, $eu = ue = u$, $um = mu = 0$ for $m \in M$. It is clear that $R^u$ is then a $k$-algebra with unit element $e$ having $R$ as a subalgebra and $ku$ as an ideal such that $(ku)^2 = ku$. Some of the properties of $R^u$ we state in

**Theorem 2.** (i) If $R$ is simple, then $R^u$ is prime.
(ii) If $R$ is semiprime, then $R^u$ is semiprime.
(iii) If $R$ is semiprime, then $R^u$ is semiprime.
(iv) If $f(x_1, \ldots, x_n)$ is a relation on $R$, then $f(x_1, \ldots, x_n, x_{n+1}) = 0$ is a relation on $R^u$.

**Proof.** Since $ku$ is an ideal the only nontrivial statements are (i) and (ii). Let $I \neq 0$ be an ideal of $R^u$ and $x = ae + \beta u + \delta m \in I$, $x \neq 0$. Then $y := x - ux = \alpha (e - u) + \delta m \in I$. In case $R \cap I = 0$ we have $ny = yn = 0$ for each $n \in M$. From $y^2 = \alpha y$ we see $I^2 \neq 0$ for $\alpha \neq 0$. In case $\alpha = 0$ we have $y = \delta m \in R \cap I = 0$ or $x = \beta u$ and $u \in I$. Thus $R \cap I = 0$, $I \neq 0$, implies $I^2 \neq 0$. If $R$ is semiprime, then $R \cap I = 0$ also implies $I^2 \neq 0$ since $R \cap I$ is a nonzero ideal of $R$. This proves (ii). If $R$ is simple, then $R \cap I = 0$ implies $I = R^u$. In case $R \cap I = 0$, $I \neq 0$, we see from $ny = yn = 0$ for $n \in M$ that $k(\alpha e + \delta m)$ is an ideal of $R$. Hence either $R = ke$, $m = 0$ or $\alpha e + \delta m = 0$. In the first case $M = 0$ which was excluded. Hence $\beta u \in I$, $\beta \neq 0$, and we see $I = ku$. Since the only ideals of $R^u$ are $0, ku, R^u$ we conclude that $R^u$ is prime.

**Corollary.** If $R$ satisfies one of the following relations, then $R^u$ satisfies the same relation.

(a) $(x, x, y) + (y, x, x) - 2(x, y, x) = 0$.
(b) $(x, y, z) + (y, z, x) + (z, x, y) = 0$.
(c) $(x, x, x) = 0$.
(d) $(x, y, x) = 0$ (flexibility).
(e) $[x, y] = 0$ (commutativity).

**Proof.** We have only to remark that as in Lemma 1 each of the relations can be rewritten by using commutators so that (iii) applies.

For simple $R$ we have $M M \subseteq M$ and we can find $m, m_i, n_i \in M$ such that $e = m + \sum m_i n_i$. This shows $\sum (m_i, n_i, u) = u$, $\sum (u, m_i, n_i) = -u$ and hence $R^u$ is not associative and not alternative for char $k \neq 2$. But by (iv) of Theorem 2 for associative $R$ the relation $(*)$ holds on $R^u$.

Now let $R \neq ke$ be a simple finite-dimensional associative algebra,
e.g. a quaternion algebra, over $k$. Then $R$ has a nonzero subspace $M$ containing all commutators such that $R = ke + M$ is direct. The corresponding algebra $R^u$ is prime, flexible, and satisfies (*) but is neither commutative nor associative. This shows that Theorem 1 cannot be extended to prime rings.

References


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