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FUNCTIONS OF DIRECT INTEGRALS OF OPERATORS[1]

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ABSTRACT. This paper contains two results. The first one is that the unitary dilation of a direct integral of linear contraction operators is the direct integral of unitary dilations. For each linear contraction operator T on a Hilbert space, consider $f(T)$ as a bounded linear operator. The second result states that if $T=\int \oplus T(s)\, d\mu(s)$ is decomposable then so is $f(T)$ and $f(T)=\int \oplus f(T(s))\, d\mu(s)$.

In this paper all operators will be bounded linear operators on separable complex Hilbert spaces. An operator is called *primary* if the von Neumann algebra it generates is a factor [**9**]. It was first shown by J. von Neumann [**10,** §20] that an arbitrary operator can be represented as a direct integral of primary operators.

In the first section of this paper we prove that the unitary dilation [**5**] of a direct integral of contraction operators is the direct integral of unitary dilations. For the discrete case, this result is due to Schreiber. He proved that the unitary dilation of a direct sum of contraction operators is the direct sum of the unitary dilations [**6**]. In the second section, we prove that if $T=\int_\Sigma \oplus T(s)\, d\mu(s)$ is a direct integral of contraction operators, then $f(T)$ defined as a bounded operator is decomposable, and $f(T)=\int_\Sigma \oplus f(T(s))\, d\mu(s)$.

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1. **The unitary dilation of a direct integral of contraction operators.** The concept of direct integral or generalized direct sum of Hilbert spaces was first introduced by von Neumann [**10**]. This subject is now incorporated in several books, most notably, [**2**, Chapter II] and [**8**, Chapter I].

The purpose of this section is to prove that the unitary dilation of a direct integral of contraction operators on a direct integral of Hilbert spaces is the direct integral of unitary dilations. For convenience, we shall refer to [**2**, Chapter II] for the definitions and results concerning the direct integral of Hilbert spaces and the direct inte-

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gral of operators. For the basic properties of the unitary dilation, which we shall use, we refer to [5, Chapter I].

Let Σ be a locally compact Hausdorff space (for our purposes a compact subset of the real line is sufficient) and μ a positive Borel measure on Σ. Let H be the direct integral of Hilbert spaces

$$H = \int_\Sigma \oplus H(s)\, d\mu(s)$$

which satisfies [2, Definition II. 1.5.3]. Whenever there is no chance of confusion we shall surpress the Σ. An operator A on H is called a *direct integral of operators* or a *decomposable* operator on H if $A = \int \oplus A(s)\, d\mu(s)$ satisfies [2, Definition II. 2.3.2]. If $\psi \in H$ and A is a decomposable operator on H, then $A\psi(s) = A(s)\psi(s)$, where $A(s)$ is an operator on $H(s)$. Let T be a contraction operator on a Hilbert space H. The operator U on the Hilbert space K is a *unitary dilation* of T if H is a closed subspace of K and $\langle T^n h_1, h_2\rangle = \langle U^n h_1, h_2\rangle$ whenever $h_1, h_2 \in H$ and n is any natural number ($\langle \cdot, \cdot\rangle$ is the inner product on K). The dilation U on K of T on H is *minimal* if $K = \text{span}\{U^n h \mid n \in Z \text{ and } h \in H\}$.

THEOREM 1. *Let $\mathcal{H} = \int \oplus H(s)\, d\mu(s)$ be a direct integral of Hilbert spaces and $T = \int_\Sigma \oplus T(s)\, d\mu(s)$ be a decomposable contraction on $\mathcal{H}$. Let $V(s)$ on $K'(s)$ be a minimal unitary dilation of $T(s)$ on $H(s)$ whenever $\|T(s)\| \leqq 1$. There exists a direct integral Hilbert space $\mathcal{K} = \int \oplus K(s)\, d\mu(s)$ and a decomposable unitary operator $U = \int \oplus U(s)\, d\mu(s)$ on $\mathcal{K}$ such that U is a minimal unitary dilation for T, $K'(s) = K(s)$ μ-a.e. and $U(s) = V(s)$ μ-a.e.*

PROOF. Since T is a contraction, there exists a set δ of μ-measure zero such that for $s \notin \delta$ we have $\|T(s)\| \leqq 1$. For $s \in \delta$ we shall define $K(s) \equiv \{0\}$ and for $s \notin E$ we shall define $K(s) \equiv K'(s)$.

First we must show that $\mathcal{K} = \int \oplus K(s)\, d\mu(s)$ exists. By [2, Definition II. 1.3.1] there exists a fundamental sequence of measurable vectors $\{\psi_n\}$ for $\mathcal{H}$. For any integer k define the functions ζ_{kn} by

$$\begin{aligned} \zeta_{kn}(s) &\equiv V^k(s)\psi_n(s), && \text{if } s \notin \delta, \\ &\equiv 0, && \text{if } s \in \delta. \end{aligned}$$

Now consider the set of all finite rational linear combinations of the set $\{\zeta_{kn}\}$. This latter set is denumerable and we shall denote it by $\{\phi_n\}$. First we claim that the scalar functions $\langle \phi_n(\cdot), \phi_m(\cdot)\rangle$ are μ-measurable. To see this we need only consider the scalar function

$$\begin{aligned}\langle \zeta_{nk}(s), \zeta_{mi}(s)\rangle &= \langle V^n(s)\psi_k(s), V^m(s)\psi_i(s)\rangle \quad \mu\text{-a.e.}\\ &= \langle V^{n-m}(s)\psi_k(s), \psi_i(s)\rangle \\ &= \langle T^{n-m}(s)\psi_k(s), \psi_i(s)\rangle,\end{aligned}$$

where we use the usual convention, $T^{-|n|}(s) = T^{*|n|}(s)$. Since T is decomposable, by [2, Proposition II. 2.1.1] the scalar functions $\langle \zeta_{nk}(\cdot), \zeta_{mi}(\cdot)\rangle$ are all μ-measurable. Therefore the functions $\langle \phi_n(\cdot), \phi_m(\cdot)\rangle$ are all μ-measurable. Next we shall show that $\{\phi_n(s)\}$ is dense in $K(s)$ μ-a.e. Let s not belong to δ. Since $V(s)$ on $K(s)$ is a minimal unitary dilation of $T(s)$ on $H(s)$, we have

$$\operatorname{span}\{V^n(s)h \mid n \in Z,\, h \in H(s)\} = K(s).$$

Since $\{\psi_n(s)\}$ is dense in $H(s)$, it follows that,

$$\{\phi_n(s)\}^- = \operatorname{span}\{\zeta_{nk}(s)\} = \operatorname{span}\{V^n(s)\psi_k(s)\} = K(s).$$

Thus the sequence $\{\phi_n\}$ satisfies the hypothesis of [2, Proposition II. 1.2.4] and it follows the $\mathcal{K} = \int \oplus K(s)\, d\mu(s)$ is defined in such a manner that $\mathcal{H} = \int \oplus H(s)\, d\mu(s)$ is a closed subspace of $\mathcal{K}$ [2, Proposition II. 1.7.9].

Finally we must show that $U = \int \oplus U(s)\, d\mu(s)$, where

$$\begin{aligned} U(s) &= V(s), &&\text{if } s \notin E,\\ &= 0, &&\text{if } s \in E,\end{aligned}$$

defines a bounded linear operator which is a minimal unitary dilation of T. From the definition of $\{\phi_n\}$ it follows that the function $s \to U(s)\phi_n(s)$ is again in the set $\{\phi_n\}$. Hence we have the measurability of $\langle U(\cdot)\phi_n(\cdot), \phi_m(\cdot)\rangle = \langle \phi_j(\cdot), \phi_m(\cdot)\rangle$ for some j. Therefore by [2, Proposition II. 2.1.1] it follows that $U: s \to U(s)$ is measurable. It is clear that U is a unitary dilation of T and the fact that U is minimal also follows easily. If ϕ is orthogonal to span$\{U^n\psi \mid n \in Z$ and $\psi \in \mathcal{H}\}$, then $\phi(s) \perp \{U^n(s)\psi_m(s) \mid n \in Z,\, m > 0\}$ μ-a.e. Since $U(s) = V(s)$ μ-a.e., $V(s)$ is a minimal dilation of $T(s)$ μ-a.e. and $\{\psi_n(s)\}$ is dense in $H(s)$ μ-a.e., we may conclude that $\phi(s) = 0$ μ-a.e. and $\phi = 0$. This completes the proof of the theorem.

As we remarked in the introduction, whenever the measure μ is discrete the direct integral decomposition becomes the usual direct sum. If we want to consider only primary decompositions, that is, decompositions of T into primary components as given by [2, II, §6], then it is not always clear when the decomposition will be discrete. However, for a large class of contractions this decomposition is always discrete. Whenever T is a completely nonunitary contraction such that $I-T^*T$ is compact, then the second author has shown that T is decomposed into the direct sum of primary contractions [**4**, Theorem 1]. Furthermore in this latter case each primary contraction is just a finite direct sum of an irreducible contraction having the same compactness property as the original operator [**4**, Theorem 3].

2. **The functional calculus.** As in §1, we let $T=\int \oplus T(s)\, d\mu(s)$ be a decomposable contraction operator on $\mathcal{H}=\int \oplus H(s)\, d\mu(s)$; and U, its minimal unitary dilation on $\mathcal{K}$. Let E be the spectral resolution of U such that $U=\int_{\Lambda(E)} \lambda\, dE(\lambda)$, where $\Lambda(\cdot)$ denotes the support of the measure E. B. Sz.-Nagy and C. Foiaş have defined $f(A)$ as a (not necessarily bounded) operator on $\mathcal{H}$ for a special class of functions [**5**]. However, for our purpose, we shall use a definition developed by Schreiber [**7**]. Let P be the canonical projection from $\mathcal{K}$ onto $\mathcal{H}$. For every borel subset α of the complex plane $\mathbf{C}$, let $F(\alpha)=PE(\alpha)P$. Then F is a positive operator-valued measure on $\mathcal{H}$ such that $\Lambda(F)=\Lambda(E)=\sigma(U)$, where $\sigma(U)$ denotes the spectrum of U [**7**]. We shall call F the *strong operator measure* of T. For every F-essentially bounded function f, $f(T)$ is an operator on $\mathcal{H}$ such that, for every pair of elements x, y in $\mathcal{H}$,

$$\langle f(T)x, y\rangle = \int_{\mathbf{C}} f(z)\, d\langle F(z)x, y\rangle.$$

PROPOSITION. *Let $T=\int \oplus T(s)\, d\mu(s)$ be a decomposable contraction operator on $\mathcal{H}=\int \oplus H(s)\, d\mu(s)$, and F be the strong operator measure of T. Denote by $\mathcal{B}$ the σ-field of all borel subsets of complex plane $\mathbf{C}$. Then F is decomposable and, for every α in $\mathcal{B}$, $F(\alpha)=\int \oplus F_s(\alpha)\, d\mu$, where F_s is the strong operator measure of $T(s)$ whenever $\|T(s)\| \leqq 1$.*

PROOF. In view of the proof of Theorem 1, we may assume without the loss of generality that $\|T(s)\| \leqq 1$ for all s in Σ. Since, for every $s\in\Sigma$, $H(s)$ is a closed subspace of $K(s)$ and $\mathcal{H}=\int \oplus H(s)\, d\mu(s)$ is a closed subspace of $\mathcal{K}=\int \oplus K(s)\, d\mu(s)$, it follows from [**2**, Lemma 3, p. 189] that the canonical projection P is decomposable and $P=\int \oplus P(s)\, d\mu(s)$, where $P(s)$ is the canonical projection from $K(s)$ onto $H(s)$. It was shown by the first author that if U is decomposable into a direct integral of unitary operators as $U=\int \oplus U(s)\, d\mu(s)$, then so is its spectral resolution E [**1**, Theorem 2.7]. In fact, for each $\alpha\in\mathcal{B}$, $E(\alpha)=\int \oplus E_s(\alpha)\, d\mu(s)$, where $E_s(\alpha)$ is the spectral resolution of $U(s)$ for each s in Σ. Thus

$$F(\alpha) = PE(\alpha)P = \int \oplus P(s)E_s(\alpha)P(s)\, d\mu(s) = \int \oplus F_s(\alpha)\, d\mu(s).$$

The proposition is proven.

Let $\mathcal{B}(\Sigma)$ denote the σ-field of Borel subsets of Σ and let δ^c denote the complement of δ in $\mathbf{C}$. If $A = \int \oplus A(s)\, d\mu(s)$ is a scalar operator in Dunford's sense [**3**], then the spectrum $\sigma(A)$ of A is

$$\bigcap \{ \{ \bigcup \sigma(A(s)) \mid s \in \delta,\ \delta \in \mathcal{B}(\Sigma)\}^- \mid \mu(\delta^c) = 0\}$$

[**1**, Lemma 2.6]. Since $\Lambda(F_s) = \Lambda(E_s) = \sigma(U(s))$, it follows that

$$\mathcal{L}_\infty(F) = \bigcup \{ \{ \bigcap \mathcal{L}_\infty(F_s) \mid s \in \delta,\ \delta \in \mathcal{B}(\Sigma)\} \mid \mu(\delta^c) = 0\}.$$

($\mathcal{L}_\infty(\cdot)$ denotes the vector space of essentially bounded scalar-valued functions.)

THEOREM 2. *Let $T = \int \oplus T(s)\, d\mu(s)$ be a decomposable contraction operator on $\mathcal{H} = \int \oplus H(s)\, d\mu(s)$; and F, the strong operator measure of T. If f is an F-essentially bounded function, then $f(T)$ is decomposable and $f(T) = \int \oplus f(T(s))\, d\mu(s)$ with $f(T(s))$ well-defined μ-a.e. Conversely, if f is F_s-essentially bounded for μ-a.a.s, then there is a μ-measurable vector field of operators $s \to g(T)(s))$ such that $g(T(s)) = f(T(s))$ μ-a.e. and $f(T) = \int \oplus g(T(s))\, d\mu(s)$.*

PROOF. Suppose that $f \in \mathcal{L}_\infty(F)$. Then it follows from the proposition and Fubini's theorem that, for any arbitrary pair of elements x and y in $\mathcal{H}$,

$$\begin{aligned}\langle f(T)x, y\rangle &= \int_{\mathbf{C}} f(z)\, d\langle F(z)x, y\rangle = \int_{\mathbf{C}} f(z) \int_\Sigma d(F_s(z)x(s), y(s))\, d\mu(s) \\ &= \int_\Sigma \int_{\mathbf{C}} f(z)\, d\langle F_s(z)x(s), y(s)\rangle\, d\mu(s) \\ &= \left\langle \left[\int_\Sigma \oplus f(T(s))\, d\mu(s) \right] x, y \right\rangle.\end{aligned}$$

This proves the first part of the theorem. Conversely, if f is F_s-essentially bounded for μ-a.a.s., then as remarked previously, f is F-essentially bounded so that $f(T)$ is well-defined. Let δ be a subset of Σ of measure zero such that, for $s \notin \delta$, f is F_s-essentially bounded. Let $s \to g(T(s))$ be such that $g(T(s)) = f(T(s))$ for $s \notin \delta$ and zero otherwise. It is clear that $s \to g(T(s))$ is μ-measurable. A similar argument as in the proof of the first part of the theorem provides the decomposability of $f(T)$ as well as the representation $f(T) = \int \oplus g(T(s))\, d\mu(s)$.

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