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A CHARACTERIZATION OF COMPACT GROUPS

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ABSTRACT. A locally compact group is compact if and only if it is of type I and has discrete spectrum (equivalently: its C^*-algebra is dual).

For separable groups, the present result is Theorem 3.4 of [1]. (In this case, the type I assumption is redundant; whether this is so in general seems to be an unsolved case of a problem of Naimark.) Upon dropping the hypothesis of separability, one would hope to avoid introducing it during the proof. We have, however, been forced to refer to [1], and indeed we do very little else.

THEOREM. *Suppose that G is a type* I *locally compact group with discrete spectrum. Then G is compact.*

PROOF. Let us show first that open subgroups of G and quotient groups of G also satisfy the hypotheses. The C^*-algebras of such groups are, respectively, sub-C^*-algebras and quotient C^*-algebras of the C^*-algebra of G. By 10.10.6 of [2], and the fact [3, Main Theorem] that a type I C^*-algebra is postliminary (so that if it is simple it is elementary), the C^*-algebra of G is dual. A sub-C^*-algebra and a quotient C^*-algebra of a dual C^*-algebra are dual. A dual C^*-algebra is type I and has discrete spectrum.

To prove the Theorem, inspection of §III of [1] shows that it is enough to prove that G has a compact open subgroup. As in the proof of Theorem 2.7 of [1], let us start with the fact that G, a locally compact group, has an open subgroup H with a compact normal subgroup K such that H/K is Lie. The connected component of the identity in H/K is open, and is separable; hence by the preceding paragraph and by Corollary 2.6 of [1] it is compact. Its inverse image in H is open and compact.

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