ON A PROBLEM OF S. ULAM

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Abstract. Giving a negative answer to a problem of S. Ulam, the author provides an example of two noncomplete subspaces of \( \mathbb{R} \) which are not isometric but whose squares are isometric.

In his book [1], S. Ulam formulated this problem:

"If \( A \) and \( B \) are metric spaces, then \( A^2 \) and \( B^2 \) may also be regarded as metric spaces, the metric of a product space \( A^2 \) being defined, for example, by \( \rho((a_1, a_2), (a_3, a_4)) = [\rho^2(a_1, a_2) + \rho^2(a_3, a_4)]^{1/2} \).

Does isometry of \( A^2 \) and \( B^2 \) imply that of \( A \) and \( B \)?"

In this paper, we will give a negative answer to this question, producing a simple example of two nonisometric spaces \( X \) and \( Y \) such that the metric products \( X^2 \) and \( Y^2 \) are isometric.

Let \( X = \mathbb{Q} \subseteq \mathbb{R} \), where \( \mathbb{Q} \) is the set of rational numbers, and let \( Y = \{ p \sqrt{2} | p \in \mathbb{Q} \} \subseteq \mathbb{R} \). \( X \) and \( Y \) are metric subspaces of \( \mathbb{R} \), and \( X \) is not isometric to \( Y \). If \( f \) were an isometry from \( X \) to \( Y \), then

\[
1 = d(0, 1) = d(f(0), f(1)) = d(p \sqrt{2}, q \sqrt{2}) = \sqrt{2} | p - q | \]

and so \( \sqrt{2} = (| p - q |)^{-1} \in \mathbb{Q} \), which is a contradiction.

However, \( X^2 \) is isometric to \( Y^2 \). Considering these two spaces as metric subspaces of \( \mathbb{R}^2 \), we study \( \theta: \mathbb{R}^2 \to \mathbb{R}^2 \) where \( \theta(x, y) = ((x - y) \sqrt{2}/2, (x + y) \sqrt{2}/2) \), the rotation by an angle of \( \pi/4 \) and therefore an isometry of \( \mathbb{R}^2 \) onto itself.

\[
\theta(X^2) = Y^2, \text{ for } \theta(X^2) \subseteq Y^2: (p, q) \in X^2 \text{ implies } \theta(p, q) = ((p - q) \sqrt{2}/2, (p + q) \sqrt{2}/2) \in Y^2, \text{ because } (p - q)/2 \text{ and } (p + q)/2 \text{ are rational; and } \theta(X^2) \supseteq Y^2: (p \sqrt{2}, q \sqrt{2}) \in Y^2 \text{ is the image of } (p + q, q - p), \text{ an element of } X^2.
\]

The contraction \( g \) of \( \theta \) to the pair \( (X^2, Y^2) \) (i.e. \( g:X^2 \to Y^2 \) where \( g(p, q) = \theta(p, q) \)) is a distance-preserving bijection, and thus an isometry from \( X^2 \) to \( Y^2 \).

Notice that, in this example, \( X \) and \( Y \) are not complete metric spaces; so the following question may be raised: if \( A \) and \( B \) are complete metric spaces, does isometry of \( A^2 \) and \( B^2 \) imply that of \( A \) and \( B \)?

References


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