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# ON A PROBLEM OF S. ULAM

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ABSTRACT. Giving a negative answer to a problem of S. Ulam, the author provides an example of two noncomplete subspaces of $\boldsymbol{R}$ which are not isometric but whose squares are isometric.

In his book [1], S. Ulam formulated this problem:

"If $A$ and $B$ are metric spaces, then $A^2$ and $B^2$ may also be regarded as metric spaces, the metric of a product space $A^2$ being defined, for example, by $\rho((a_1, a_2), (a_3, a_4)) = [\rho^2(a_1, a_3)+\rho^2(a_2, a_4)]^{1/2}$.

Does isometry of $A^2$ and $B^2$ imply that of $A$ and $B$?"

In this paper, we will give a negative answer to this question, producing a simple example of two nonisometric spaces $X$ and $Y$ such that the metric products $X^2$ and $Y^2$ are isometric.

Let $X=\boldsymbol{Q}\subseteq\boldsymbol{R}$, where $\boldsymbol{Q}$ is the set of rational numbers, and let $Y=\{p\sqrt{2}\mid p\in\boldsymbol{Q}\}\subseteq\boldsymbol{R}$. $X$ and $Y$ are metric subspaces of $\boldsymbol{R}$, and $X$ is not isometric to $Y$. If $f$ were an isometry from $X$ to $Y$, then

$$1 = d(0, 1) = d(f(0), f(1)) = d(p\sqrt{2}, q\sqrt{2}) = \sqrt{2}\,|\,p - q\,|$$

and so $\sqrt{2} = (|\,p-q\,|)^{-1}\in\boldsymbol{Q}$, which is a contradiction.

However, $X^2$ is isometric to $Y^2$. Considering these two spaces as metric subspaces of $\boldsymbol{R}^2$, we study $\theta:\boldsymbol{R}^2\rightarrow\boldsymbol{R}^2$ where $\theta(x, y) =((x-y)\sqrt{2}/2, (x+y)\sqrt{2}/2)$, the rotation by an angle of $\pi/4$ and therefore an isometry of $\boldsymbol{R}^2$ onto itself.

$\theta(X^2)=Y^2$, for $\theta(X^2)\subseteq Y^2$: $(p, q)\in X^2$ implies $\theta(p, q)=((p-q)\sqrt{2}/2, (p+q)\sqrt{2}/2)\in Y^2$, because $(p-q)/2$ and $(p+q)/2$ are rational; and $\theta(X^2)\supseteq Y^2$: $(p\sqrt{2}, q\sqrt{2})\in Y^2$ is the image of $(p+q, q-p)$, an element of $X^2$.

The contraction $g$ of $\theta$ to the pair $(X^2, Y^2)$ (i.e. $g:X^2\rightarrow Y^2$ where $g(p, q)=\theta(p, q)$) is a distance-preserving bijection, and thus an isometry from $X^2$ to $Y^2$.

Notice that, in this example, $X$ and $Y$ are not complete metric spaces; so the following question may be raised: if $A$ and $B$ are complete metric spaces, does isometry of $A^2$ and $B^2$ imply that of $A$ and $B$?

## REFERENCES

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