A ZERO-ONE LAW FOR GAUSSIAN PROCESSES

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Abstract. Let $P_0$ be a Gaussian probability measure on the measurable space $(X, B(X))$, where $X$ is a linear space of real-valued functions over a complete separable metric space $T$, and $B(X)$ is the $\sigma$-algebra generated by sets of the form \( \{ x \in X : (x(t_1), \ldots, x(t_n)) \in B^n \} \); $B^n$ being the Borel sets of $\mathbb{R}^n$, $n \geq 1$. The covariance $R(s, t)$ is assumed continuous on $T \times T$. If $G$ is a subgroup of $X$ and belongs to the $\sigma$-algebra $B_0(X)$ (the completion of $B(X)$ with respect to $P_0$), then it is shown that $P_0(G) = 0$ or 1.

1. Introduction. For the background and history of the problem we refer to [1] and [2]. We would like to point out here that Jamison and Orey in [1] proved the above result for the special case where the Gaussian process involved had continuous paths. Kallianpur [2] proved such a result for $r$-modules (groups closed under multiplication by rationals). Kallianpur's result for groups is restricted to those which are $B(X)$-measurable rather than $B_0(X)$-measurable. He points out in [2] why his proof does not work for $B_0(X)$-measurable subgroups. Our main result (Theorem 1) unifies and generalizes the results of [1] and [2], and also gives an answer in the affirmative to the conjecture made in [1]. Our method of proof is similar to the one given in [2], but it is simpler. The notation used here is also essentially the same as in [2].

$T$ is a complete separable metric space. $X$ is a linear space of real-valued functions defined on $T$ with the usual operation of addition of functions and multiplication by real scalars. $B(X)$ is the $\sigma$-algebra as explained above. $P_0$ is a Gaussian measure on $(X, B(X))$. We assume that

\begin{align}
\int_X x(t) P_0(dx) &= 0 \quad \text{for each } t \in T; \\
\int_X x(s)x(t) P_0(dx) &= R(s, t), \quad s, t \in T,
\end{align}

is continuous on $T \times T$. $H(R)$ will denote the reproducing kernel Hilbert space of the covariance $R$. It is also assumed that $H(R) \subset X$. 

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$B_0(X)$ denotes the completion of the $\sigma$-algebra $B(X)$ with respect to $P_0$.

The main result is the following:

**Theorem 1.** Let $G$ be a $B_0(X)$-measurable subgroup of the linear space $X$. Then $P_0(G) = 0$ or 1.

**Remark.** For applications of this result we refer to Lemma 5 [2].

2. **Proof of Theorem 1.** We need the following lemmas.

**Lemma 1.** Let $F \in B_0(X)$. If $m \in H(R)$ and $\alpha$ is a real number, then the set

$$F_{\alpha} = \{ x \in X : x = y + \alpha m, y \in F \}$$

is in $B_0(X)$, and

$$\lim_{\alpha \to 0} P_0(F_{\alpha}) = P_0(F).$$

**Proof.** This is established in the proof of Lemma 5 [2].

**Lemma 2.** Let $\{e_j\}$ be a complete orthonormal system in $H(R)$ and $g$ be a $B_0(X)$-measurable real function such that for each $x \in X$ and every rational $r \in H(X)$, $g(x + re_j) = g(x)$, $j = 1, 2, \ldots$. Then $g(x) = \text{constant a.s. (P_0)}$.

**Proof.** See Lemma 6 [2].

To finish the proof of the theorem, let $P_0(G) > 0$. We will show that this implies that $H(R) \subseteq G$. It is then seen easily that $P_0(G) = 1$ as follows: since $G$ is a group and $H(R)$ a Hilbert space with a complete orthonormal system $\{e_j\}$, it follows that $x \in G$ if and only if $x + re_j \in G$ for every rational $r$, $j = 1, 2, \ldots$. Let $I_0$ be the indicator function of the set $G$, then by Lemma 2 we have $I_0 = \text{constant a.s. (P_0)}$. But this constant must be equal to 1 since $P_0(G) > 0$.

It thus remains to show that $P_0(G) > 0$ implies that $H(R) \subseteq G$. There exists a positive integer $s$ such that $P_0(G) > 1/s$. We hold this $s$ fixed. Let $m \in H(R)$, $m \in G$. This will lead to a contradiction. For each positive integer $n$ define $s + 1$ sets $G_0^{(n)}, G_1^{(n)}, \ldots, G_s^{(n)}$ as follows:

$$G_0^{(n)} = G, \quad G_k^{(n)} = \{ x : x = y + (s!kn)^{-1}m, y \in G \}, 1 \leq k \leq s. \quad (2.2)$$

By Lemma 1 these sets are in $B_0(X)$ and

$$\lim_{n} P_0(G_k^{(n)}) = P_0(G), \quad 0 \leq k \leq s. \quad (2.3)$$

We now show that these sets must be pairwise disjoint. If $G_0^{(n)}$ and
$G^{(n)}, k > 0$, have an element in common, then for some $x, y \in G$, $x = y + (s!kn)^{-1}m$; hence $m = (s!kn)(x - y)$, an element of $G$, which contradicts that $m \in G$. Suppose now that $G^{(n)}_j, G^{(n)}_k, 1 \leq j < k$, have an element in common. Then for some $x, y \in G$,

$$x + (s!jn)^{-1}m = y + (s!kn)^{-1}m.$$ 

Hence $m = (s!kn)(k - j)^{-1}(y - x)$. But $(s!kn) (k - j)^{-1}$ is a positive integer and this implies again that $m \in G$, a contradiction. We thus have $\sum_{n=0}^\infty P_0(G^{(n)}) \leq 1, n = 1, 2, \cdots$. Letting $n$ tend to infinity, we conclude from (2.3) that $(s+1)P_0(G) \leq 1$. Since $P_0(G) > 1/s$, this is a contradiction. Hence $H(R) \subseteq G$ and the proof is complete.

References


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