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A ZERO-ONE LAW FOR GAUSSIAN PROCESSES

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ABSTRACT. Let P_0 be a Gaussian probability measure on the measurable space $(X, B(X))$, where X is a linear space of real-valued functions over a complete separable metric space T, and $B(X)$ is the σ-algebra generated by sets of the form $\{x \in X : (x(t_1), \cdots, x(t_n)) \in B^n\}$; B^n being the Borel sets of R^n, $n \geqq 1$. The covariance $R(s, t)$ is assumed continuous on $T \times T$. If G is a subgroup of X and belongs to the σ-algebra $B_0(X)$ (the completion of $B(X)$ with respect to P_0), then it is shown that $P_0(G) = 0$ or 1.

1. **Introduction.** For the background and history of the problem we refer to [1] and [2]. We would like to point out here that Jamison and Orey in [1] proved the above result for the special case where the Gaussian process involved had continuous paths. Kallianpur [2] proved such a result for r-modules (groups closed under multiplication by rationals). Kallianpur's result for groups is restricted to those which are $B(X)$-measurable rather than $B_0(X)$-measurable. He points out in [2] why his proof does not work for $B_0(X)$-measurable subgroups. Our main result (Theorem 1) unifies and generalizes the results of [1] and [2], and also gives an answer in the affirmative to the conjecture made in [1]. Our method of proof is similar to the one given in [2], but it is simpler. The notation used here is also essentially the same as in [2].

T is a complete separable metric space. X is a linear space of real-valued functions defined on T with the usual operation of addition of functions and multiplication by real scalars. $B(X)$ is the σ-algebra as explained above. P_0 is a Gaussian measure on $(X, B(X))$. We assume that

$$\int_X x(t) P_0(dx) = 0 \quad \text{for each } t \in T; \tag{1.1}$$

$$\int_X x(s) x(t) P_0(dx) = R(s, t), \qquad s, t \in T, \tag{1.2}$$

is continuous on $T \times T$. $H(R)$ will denote the reproducing kernel Hilbert space of the covariance R. It is also assumed that $H(R) \subset X$.

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$B_0(X)$ denotes the completion of the σ-algebra $B(X)$ with respect to P_0.

The main result is the following:

THEOREM 1. *Let G be a $B_0(X)$-measurable subgroup of the linear space X. Then $P_0(G)=0$ or 1.*

REMARK. For applications of this result we refer to Lemma 5 [2].

2. **Proof of Theorem 1.** We need the following lemmas.

LEMMA 1. *Let $F\in B_0(X)$. If $m\in H(R)$ and α is a real number, then the set*

$$F_\alpha = \{x \in X : x = y + \alpha m,\ y \in F\}$$

is in $B_0(X)$, and

$$\lim_{\alpha\to 0} P_0(F_\alpha) = P_0(F). \tag{2.1}$$

PROOF. This is established in the proof of Lemma 5 [2].

LEMMA 2. *Let $\{e_j\}_1^\infty$ be a complete orthonormal system in $H(R)$ and g be a $B_0(X)$-measurable real function such that for each $x\in X$ and every rational r $g(x+re_j)=g(x)$, $j=1, 2, \cdots$. Then $g(x)=$ constant a.s. (P_0).*

PROOF. See Lemma 6 [2].

To finish the proof of the theorem, let $P_0(G)>0$. We will show that this implies that $H(R)\subset G$. It is then seen easily that $P_0(G)=1$ as follows: since G is a group and $H(R)$ a Hilbert space with a complete orthonormal system $\{e_j\}$, it follows that $x\in G$ if and only if $x+re_j\in G$ for every rational r, $j=1, 2, \cdots$. Let I_G be the indicator function of the set G, then by Lemma 2 we have $I_G=$constant a.s. (P_0). But this constant must be equal to 1 since $P_0(G)>0$.

It thus remains to show that $P_0(G)>0$ implies that $H(R)\subset G$. There exists a positive integer s such that $P_0(G)>1/s$. We hold this s fixed. Let $m\in H(R)$, $m\notin G$. This will lead to a contradiction. For each positive integer n define $s+1$ sets $G_0^{(n)}, G_1^{(n)}, \cdots, G_s^{(n)}$ as follows:

$$G_0^{(n)} = G,\quad G_k^{(n)} = \{x : x = y + (s!kn)^{-1}m,\ y \in G\},\ 1 \leqq k \leqq s\}. \tag{2.2}$$

By Lemma 1 these sets are in $B_0(X)$ and

$$\lim_n P_0(G_k^{(n)}) = P_0(G), \qquad 0 \leqq k \leqq s. \tag{2.3}$$

We now show that these sets must be pairwise disjoint. If $G_0^{(n)}$ and

$G_k^{(n)}$, $k>0$, have an element in common, then for some x, $y \in G$, $x = y + (s!kn)^{-1}m$; hence $m = (s!\,kn)(x-y)$, an element of G, which contradicts that $m \notin G$. Suppose now that $G_j^{(n)}$, $G_k^{(n)}$, $1 \leqq j < k$, have an element in common. Then for some $x, y \in G$,

$$x + (s!jn)^{-1}m = y + (s!kn)^{-1}m.$$

Hence $m = (s!\,kjn)(k-j)^{-1}(y-x)$. But $(s!\,kjn)\ (k-j)^{-1}$ is a positive integer and this implies again that $m \in G$, a contradiction. We thus have $\sum_{k=0}^{s} P_0(G_k^{(n)}) \leqq 1$, $n = 1, 2, \cdots$. Letting n tend to infinity, we conclude from (2.3) that $(s+1)P_0(G) \leqq 1$. Since $P_0(G) > 1/s$, this is a contradiction. Hence $H(R) \subset G$ and the proof is complete.

References

1. B. Jamison and S. Orey, *Subgroups of sequences and paths*, Proc. Amer. Math. Soc. **24** (1970), 739–744. MR **40** #8121.

2. G. Kallianpur, *Zero-one laws for Gaussian processes*, Trans. Amer. Math. Soc. **149** (1970), 199–211.

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