MUTUAL ABSOLUTE CONTINUITY OF
SETS OF MEASURES

BERTRAM WALSH

Abstract. A theorem slightly stronger than the following is proved: If \( K \) is a convex set of (signed) measures that are absolutely continuous with respect to some fixed positive sigma-finite measure, then the subset consisting of those measures in \( K \) with respect to which all measures in \( K \) are absolutely continuous is the complement of a set of first category in any topology finer than the norm topology of measures. This implies, e.g., that any Banach-space-valued measure \( \mu \) is absolutely continuous with respect to \( |\langle \mu(\cdot), x'\rangle| \) for a norm-dense \( G_1 \) of elements \( x' \) of the dual of the Banach space.

Recently Rybakov [4] proved, by a direct but rather involved construction, that if \((S, \Sigma)\) is a measurable space and \( \mu: \Sigma \to E \) a countably additive measure with values in a Banach space \( E \), then there exists a linear functional \( x'_0 \in E' \) such that \( \mu \) is absolutely continuous with respect to the (variation of the) scalar-valued measure \( \langle \mu(\cdot), x'_0 \rangle \). This note gives a rather less involved proof of a stronger result, as indicated in the abstract above.

To fix our ideas and notation, let \((S, \Sigma, \lambda)\) be a \( \sigma \)-finite measure space; since there is a strictly positive function in \( L^1(\lambda) \), there is no loss of generality in taking \( \lambda \) finite, which we shall do. We shall denote by \( P_n \) \((n = 1, 2, \ldots, \infty)\) the set of all \( n \)-tuples \( \alpha = (\alpha_1, \alpha_2, \ldots) \) in \( R^n \) (or \( l^\infty \) if \( n = \infty \)) for which all \( \alpha_i > 0 \), and by \( Q_n \) \((n = 1, 2, \ldots, \infty)\) the subset of \( P_n \) for which \( \sum \alpha_i = 1 \). It is obvious that the \( P_n \)'s and \( Q_n \)'s are \( G_1 \)'s in their ambient spaces \( R^n \) and \( l^\infty \), and thus their topologies can be generated by complete metrics; in particular, they are Baire spaces (spaces for which the Baire category theorem holds).

Any measure-theoretic terms or notation not otherwise explained agree with the usage of [1]—except that \( |\mu| \) is the variation of \( \mu \).

Lemma 1. For any set \( C \subseteq \alpha(S, \Sigma) \) of measures absolutely continuous with respect to \( \lambda \), there exists a countable set \( \{\mu_i\}_{i \in I} \subseteq C \) such that if \( |\mu_i|(A) = 0 \) for all \( i \), then \( |\mu|(A) = 0 \) for all \( \mu \subseteq C \).

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Proof. Assuming $C \neq \emptyset$, one can find sets $A \in \Sigma$ for which there exist elements $\mu \in C$ such that $|\mu|(A) > 0$ and every $|\mu|$-null subset of $A$ is $\lambda$-null; e.g., if $\mu \neq 0$ and $f$ is its Radon-Nikodým derivative with respect to $\lambda$, then $A = \{s \in S : |f(s)| \neq 0\}$ has that property. Zorn's lemma gives a maximal disjoint family $\{A_i\}_{i \in I}$ of such sets, with corresponding measures $\{\mu_i\}_{i \in I}$. Since all but countably many $A_i$'s must be $\lambda$-null, $I$ is countable; the existence of an $A \in \Sigma$ with $|\mu_i|(A) = 0$ for all $i \in I$ but $|\mu|(A) > 0$ for some $\mu \in C$ would clearly contradict maximality. Q.E.D.

Let $\mathfrak{M}$ denote the topological vector space of all $\Sigma$-measurable scalar-valued functions on $\mathcal{S}$ under the topology of convergence in $\lambda$-measure. It is easy to verify that for any $A \in \Sigma$, $\delta > 0$ and open set $U$ of scalars, the set $\{f \in \mathfrak{M} : \lambda(\{s \in A : f(s) \in U\}) < \delta\}$ is an open set in $\mathfrak{M}$, and (consequently) the set $\{f \in \mathfrak{M} : \lambda(\{s \in A : f(s) \not\in U\}) = 0\}$ is a $G_\delta$ in $\mathfrak{M}$.

Lemma 2. Let $\{f_i\}_{i=1}^\infty$ be a sequence in $\mathfrak{M}$, and set $B_i = \{s \in S : f_i(s) \neq 0\}$. Then

1. for any finite $n$, the set of $\alpha \in P_n$ for which

$$\lambda\left(\left\{s \in \bigcup_{i=1}^n B_i : \sum_{i=1}^n |\alpha f_i(s)| = 0\right\}\right) = 0$$

is a dense $G_\delta$ in $P_n$ or $Q_n$ respectively; moreover,

2. the set of $\alpha \in Q_\infty$ for which

$$\lambda\left(\left\{s \in \bigcup_{i=1}^n B_i : \sum_{i=1}^n |\alpha f_i(s)| = 0\right\}\right) = 0$$

for all finite $n$ is a dense $G_\delta$ in $Q_\infty$.

Proof. For any finite $n$, the map $(\alpha_1, \alpha_2, \ldots) \mapsto \sum_{i=1}^n \alpha f_i$ is continuous from $R^n$ or $l^n_R$ to $\mathfrak{M}$, and thus all the sets in question are $G_\delta$'s. To prove the "density" part of assertion (1) for $Q_n$ it clearly suffices to prove the part concerning $P_n$, and because $P_n = P_{n-1} \times P_1$ a proof by induction on $n$ follows immediately from verification of the case $n = 2$. To check that case we need only observe that the sets $S_\beta = \{s \in B_1 \cup B_2 : f_1(s) + \beta f_2(s) = 0\}$ are disjoint for distinct $\beta \in P_1$, and must therefore all be $\lambda$-null except for countably many choices of $\beta$. It follows that if $(\alpha_1, \alpha_2)$ does not belong to the nowhere-dense set of solutions of $\alpha_2 = \beta \alpha_1$ for one of those countably many $\beta$'s, then

$$\lambda(\{s \in B_1 \cup B_2 : \alpha_1 f_1(s) + \alpha_2 f_2(s) = 0\}) = 0;$$
so the set of \((\alpha_1, \alpha_2)\)'s for which that relation holds contains the complement of a set of first category and is thus dense.

From assertion (1) it follows immediately that for any fixed finite \(n\) the set

\[
\{ \alpha \in Q_w \lambda \left( \left\{ s \in \bigcup_{i=1}^{n} B_i : \sum_{i=1}^{n} \alpha f_i(s) = 0 \right\} \right) = 0 \}
\]

is a dense \(G_\delta\) in \(Q_w\). The set of \(\alpha \in Q_w\) for which that relation holds for all finite \(n\) is thus a countable intersection of dense \(G_\delta\)'s in \(Q_w\) and thus again dense by the Baire category theorem, a fact which establishes (2). Q.E.D.

We can now prove the principal theorem, stated in somewhat more general form than given in the abstract above.

**Theorem.** Let \(K\) be a bounded\(^2\) convex set in a locally convex space \(F\), and suppose \(K\) is a Baire space. Let \(T: K \to \text{ca}(S, \Sigma)\) be an affine map continuous in the norm topology of \(\text{ca}(S, \Sigma)\), and suppose that the measures in \(T[K]\) are all absolutely continuous with respect to \(\lambda\). Then the set of points \(x \in K\) for which all measures in \(T[K]\) are absolutely continuous with respect to \(|Tx|\) is a dense \(G_\delta\) in \(K\).

**Proof.** Setting \(C = T[K]\) in Lemma 1, we can select a sequence \(\{x_i\}_{i=2}^{\infty}\) in \(K\) which has the property that if \(\mu_i = Tx_i\), then \(|\mu_i|(A) = 0\) for all \(i\) implies \(|Tx|(A) = 0\) for all \(x \in K\). Let \(f_i\) be a Radon-Nikodým derivative of \(\mu_i\) with respect to \(\lambda\), \(i = 2, 3, \ldots\), let \(B_i = \{s \in S : f_i(s) \neq 0\}\), and let \(B = \bigcup_{i=2}^{\infty} B_i\). For each \(\epsilon > 0\), the set

\[
W_{\epsilon} = \{\nu \in \text{ca}(S, \Sigma) : \text{there exists } \delta > 0 \text{ such that } |\nu|(A) < \delta \implies \lambda(A \cap B) < \epsilon \}
\]

is open in the norm topology of \(\text{ca}(S, \Sigma)\); indeed, if \(\delta\) has that property for \(\nu\) and \(|\nu - \pi| < \delta/2\), then \(\delta/2\) has that property for \(\pi\). Thus \(T^{-1}[W_{\epsilon}] = \{x \in K : \text{there exists } \delta > 0 \text{ such that } |Tx|(A) < \delta \implies \lambda(A \cap B) < \epsilon\}\) is open in \(K\), and we claim it is dense. To see this, let a point \(x_1 \in K\) and a convex zero-neighborhood \(N\) in \(F\) be given, and let \(f_1\) be a Radon-Nikodým derivative of \(\mu_1 = Tx_1\); set \(B_1 = \{s \in S : f_1(s) \neq 0\}\). Let \(\eta > 0\) be so small that \(0 < \eta \implies (1 - t)x_1 + tK \subseteq x_1 + N\); this is possible since \(K\) is bounded. Applying Lemma 2 to the sequence \(\{f_i\}_{i=1}^{\infty}\), we can find an \(\alpha \in Q_w\) satisfying condition (2) of that lemma and such that \(\sum_{i=2}^{n} \alpha_i < \eta\). If \(n\) is so large that \((\sum_{i=1}^{n} \alpha_i)^{-1}(\sum_{i=2}^{n} \alpha_i) < \eta\) and \(\lambda(B \cup B_2 \cup B_i) < \epsilon/2\), then setting \(\gamma_i\)

\(^2\) Alternatives to this hypothesis are possible. E.g., the assertion of the theorem holds if \(K\) is metrizable and \(0 \in K\), even if \(K\) is unbounded.

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\left(\sum_{i=1}^{n} \alpha_i\right)^{-1} \alpha_i = \sum_{i=1}^{n} \gamma_i x_i \in K \cap (x_i + N), \quad \text{and} \quad T(\sum_{i=1}^{n} \gamma_i x_i) = \sum_{i=1}^{n} \gamma_i x_i \text{ which has Radon-Nikodým derivative } \sum_{i=1}^{n} \gamma_i f_i \text{ with respect to } \lambda. \text{ Since }
\]

\[
\lambda\left(\left\{s \in \bigcup_{i=1}^{n} B_i : \sum_{i=1}^{n} \gamma_i f_i(s) = 0\right\}\right) = 0
\]

(for the \(\gamma_i\)'s are proportional to the \(\alpha_i\)'s), one sees that \(\lambda\) is absolutely continuous with respect to \(\sum_{i=1}^{n} \gamma_i x_i\) on \(\bigcup_{i=2}^{n} B_i\), i.e., if \(H \subseteq \bigcup_{i=2}^{n} B_i\) then \(\lambda(H)\) can be made arbitrarily small by taking \(\left|\sum_{i=1}^{n} \gamma_i x_i\right|(H)\) sufficiently small. In particular, there exists \(\delta > 0\) with the property that for such an \(H\), \(\left|\sum_{i=1}^{n} \gamma_i x_i\right|(H) < \delta = \lambda(H) < \epsilon/2\); so if \(A \subseteq \Sigma\) is such that \(\left|\sum_{i=1}^{n} \gamma_i x_i\right|(A) < \delta\), then

\[
\lambda\left(A \cap \bigcup_{i=2}^{n} B_i\right) < \epsilon/2 \quad \text{and} \quad \lambda\left(A \cap \left(B \setminus \bigcup_{i=2}^{n} B_i\right)\right) < \epsilon/2
\]

and thus \(\lambda(A \cap B) < \epsilon\).

We now need only use the fact that \(K\) is a Baire space to conclude that \(K_0 = T^{-1}[\bigcap_{k=1}^{n} W_{1/k}]\) is dense in \(K\). It is immediate that \(K_0\) is the set of all \(x \in K\) such that \(\left|Tx\right|(A) = 0\) implies \(\lambda(A \cap B) = 0\); but since \(S \setminus B\) is null with respect to all \(\mu\) and thus with respect to \(\left|Ty\right|\) for all \(y \in K\), \(K_0\) is the set of all \(x \in K\) such that \(\left|Tx\right|(A) = 0\Rightarrow\left|Ty\right|(A) = 0\) for all \(y \in K\). Q.E.D.

A strengthened form of Rybakov's theorem follows at once. Indeed, if \(\mu: \Sigma \to E\) is a Banach-space valued vector measure, there is an adjoint linear mapping \(T: E' \to ca(S, \Sigma)\) given by \((Tx')(A) = \langle \mu(A), x' \rangle\) and \(T\) is continuous in the norm topologies. It is known that there exists a finite positive measure \(\lambda \in ca(S, \Sigma)\) with respect to which all the measures \(Tx'\) are absolutely continuous (see [1, IV.10.5, p. 321] or, for a direct, elegant proof see [2, Theorem 3.10, p. 199 ff.]). Taking the set \(K\) of our theorem to be the closed unit ball in \(E'\) (with the norm topology) and \(T\) to be the mapping just described, we see that the set \(K_0\) of \(x_0\) in that unit ball having the property that \(\mu\) is absolutely continuous with respect to \(\langle \mu(\cdot), x_0 \rangle\) is a norm-dense \(G_\delta\). This has the amusing consequence that the norm of elements \(x \in E\) can be computed by taking the supremum of \(\left|\langle x, x_0 \rangle\right|\) over \(x_0 \in K_0\). For another application of the theorem, suppose \(X\) is a simplex in the sense of Choquet [3], and let \(P\) be the unique linear extension of the mapping which assigns to each positive measure \(\mu\) on \(X\) the maximal measure which represents \(\mu\). \(P\) is norm-decreasing and therefore norm-continuous. Thus if \(K\) is a norm-closed convex set of measures on \(X\) such that \(P[K]\) is ab-
solutely continuous with respect to some fixed positive measure \( \lambda \), then there is a norm-dense \( G_\delta \) set \( K_\delta \subseteq K \) such that every measure in \( P[K] \) is absolutely continuous with respect to \( |P\mu| \) for each \( \mu \in K_\delta \).

**References**


