A WHITEHEAD TYPE THEOREM

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Abstract. Let $\mathcal{F}$ denote the Serre class of finite abelian groups. We consider, for example, conditions under which a map which induces an $\mathcal{F}$-epimorphism in homotopy also induces an $\mathcal{F}$-epimorphism in homology.

1. Introduction. Let $f: X \to Y$ be a map, $\mathcal{C}$ a Serre class of abelian groups. Modulo some technical assumptions on the spaces $X$ and $Y$ or the class $\mathcal{C}$, the Whitehead theorem states that $f$ induces a $\mathcal{C}$-isomorphism in homotopy in each dimension if and only if it induces a $\mathcal{C}$-isomorphism in homology in each dimension. We are concerned here with finding conditions under which the word “isomorphism” can be replaced by “epimorphism” or “monomorphism” for the class $\mathcal{F}$ of finite abelian groups. For example, we show that if a map $g$ from an $H$-space $Y$ to a 1-connected finite CW-complex $X$ induces an $\mathcal{F}$-epimorphism in homotopy, then it induces an $\mathcal{F}$-epimorphism in homology and $X$ is an $H$-space mod $\mathcal{F}$. As a special case we recover a result of [4]: If $X$ is a 1-connected finite CW-complex and a $G$-space mod $\mathcal{F}$ (i.e. the evaluation map $\omega: (X^2, 1) \to (X, *)$ induces an $\mathcal{F}$-epimorphism in homotopy), then $X$ is an $H$-space mod $\mathcal{F}$. (The converse is also true.) Moreover, $\omega$ induces an $\mathcal{F}$-epimorphism in homology. The proof given here is much simpler than that given in [4].

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All spaces are assumed to have the based homotopy type of a CW-complex and all maps and homotopies are to preserve base points. We will frequently not distinguish between a map and its homotopy class. The symbol “$\mathcal{F}$” will be used to denote Hurewicz homomorphisms. We assume that the reader is familiar with [1] and [2].

2. The result. Let $X$ be a 1-connected finite CW-complex, $Y$ be a 1-connected space with $H_m(Y)$ finitely generated for all $m$ and let $f: X \to Y, g: Y \to X$ be maps.

Theorem 1. Suppose that $Y$ is an $H$-space.
(i) If $f_*: \pi_m(X) \to \pi_m(Y)$ is an $\mathfrak{f}$-monomorphism for all $m$, then so is $f_*: H_m(X) \to H_m(Y)$.

(ii) If $g_*: \pi_m(Y) \to \pi_m(X)$ is an $\mathfrak{f}$-epimorphism for all $m$, then so is $g_*: H_m(Y) \to H_m(X)$. Moreover, $X$, in each case, is an $H$-space mod $\mathfrak{f}$.

**Theorem 2.** Suppose that $Y$ is an $H'$-space.

(i) If $f_*: H_m(X) \to H_m(Y)$ is an $\mathfrak{f}$-monomorphism for all $m$, then so is $f_*: \pi_m(X) \to \pi_m(Y)$.

(ii) If $g_*: H_m(Y) \to H_m(X)$ is an $\mathfrak{f}$-epimorphism for all $m$, then so is $g_*: \pi_m(Y) \to \pi_m(X)$. Moreover, $X$, in each case, is an $H'$-space mod $\mathfrak{f}$.

**Remark.** The 1-connectedness assumption on $Y$ is needed only for Theorem 2 (ii) and neither assumption on $Y$ is needed for Theorem 1 (ii).

We will need the following result, the proof of which depends only on the universal coefficient theorem and the representability of cohomology.

**Lemma 1.** Let $B$ be a space for which $H_m(B)$ is finitely generated and let $\beta \in \pi_m(B)$. Then $h(\beta) \neq 0$ if and only if there is a map $h: B \to K(\pi, m)$ (= an Eilenberg-Mac Lane space) such that $h\beta$ is not homotopic to a constant. (We may take $\pi = \mathbb{Z}_p$ or $\mathbb{Z}$ depending on whether $h(\beta)$ has finite or infinite order.)

**Lemma 2.** Let $a: A \to B$ be a map from an $H$-space $A$ to a finite CW-complex $B$. If $a_*: \pi_{2n}(A) \to \pi_{2n}(B)$ is an $\mathfrak{f}$-epimorphism, then $h(\pi_{2n}(B)) \in \mathfrak{f}$.

**Proof.** It suffices to show that $h(\beta)$ has finite order for each $\beta \in \pi_{2n}(B)$. In order to obtain a contradiction, assume that there is a $\beta \in \pi_{2n}(B)$ such that $h(\beta)$ has infinite order. By Lemma 1, there is a map $h: B \to K(Z, 2n)$ such that $h\beta$ is not homotopic to a constant. Since $a_*: \pi_{2n}(A) \to \pi_{2n}(B)$ is an $\mathfrak{f}$-epimorphism there is an $\alpha \in \pi_{2n}(A)$ such that $a_*(\alpha) = r\beta$ where $r$ is some nonzero integer. Now, if $\rho: \Sigma A \to A$ is a retraction map ($A$ is an $H$-space), then $h \circ \Omega \Sigma \alpha: \Omega \Sigma S^{2n} \to K(Z, 2n)$ is a nontrivial map which factors through a finite complex. This is clearly impossible (consider the ring structure of $H^*(\Omega \Sigma S^{2n})$) and the lemma is proved.

**Corollary 1.** If $B$ is $(2n-1)$-connected, then $\pi_{2n}(B) \in \mathfrak{f}$.

**Proof of Theorem 1.** (i) Since the homotopy suspension homomorphism for an $H$-space is a monomorphism in all dimensions, it follows that the suspension homomorphism $i_*: \pi_m(X) \to \pi_m(\Sigma X)$ ($i$ is the inclusion map) is an $\mathfrak{f}$-monomorphism for all $m$ and therefore that $X$ is an $H$-space mod $\mathfrak{f}$. Let $h: S \to X$ be a weak $\mathfrak{f}$-equivalence,
where $S$ is a finite product of odd dimensional spheres. Since the Hurewicz homomorphism $h: \pi_m(Y) \to H_m(Y)$ is an $\mathfrak{F}$-isomorphism, it follows from Lemma 1 that $fh$ induces an $\mathfrak{F}$-epimorphism in homology and hence an $\mathfrak{F}$-monomorphism in homology. Thus $f$ induces an $\mathfrak{F}$-monomorphism in homology.

(ii) We first show, by induction, that $\pi_{2n}(X) \in \mathfrak{F}$ for all $n$. For $n = 1$, this follows from Corollary 1. Assume that $\pi_{2n}(X) \in \mathfrak{F}$ for $2n < N$, $N$ odd. Since $g^*: \pi_m(Y) \to \pi_m(X)$ is an $\mathfrak{F}$-epimorphism for all $m$, we can use the multiplication on $Y$ to obtain a map $h_N: S \to Y$ such that $gh_N$ induces an $\mathfrak{F}$-isomorphism in homotopy in dimensions $\leq N$, where $S$ is a finite product of odd dimensional spheres $S^n$, $3 \leq n_i \leq N$. We can assume that $gh_N$ is an inclusion map. Then $\pi_m(X, S) \in \mathfrak{F}$ for all $m \leq N$; by the Hurewicz theorem, $h: \pi_{2n+1}(X, S) \to H_{2n+1}(X, S)$ is an $\mathfrak{F}$-isomorphism. Since $\pi_{2n+1}(S) \in \mathfrak{F}$, it follows that $h: \pi_{2n+1}(X) \to H_{2n+1}(X)$ is an $\mathfrak{F}$-monomorphism and so, by Lemma 2, $\pi_{2n}(X) \in \mathfrak{F}$ for all $n$. It is now a simple matter to show that for $N \geq \dim X$, $gh_N$ is a weak $\mathfrak{F}$-equivalence. Therefore $X$ is an $\mathfrak{H}$-space mod $\mathfrak{F}$ and $g^*: \pi_m(F) \to \pi_m(X)$ is an $\mathfrak{F}$-epimorphism for all $m$.

Proof of Theorem 2. (i) Since $f^*: H_m(X) \to H_m(Y)$ is an $\mathfrak{F}$-monomorphism for all $m$, $f^*: H^n(Y) \to H^n(X)$ is an $\mathfrak{F}$-epimorphism for all $m$. Let $\{\beta_i\}$ be a basis for the free part of $H^*(X)$ and let $\{\gamma_i\} \subset H^*(Y)$ be chosen so that $f^*(\gamma_i) = \iota_i \beta_i$ for some nonzero integer $\iota_i$. Let $r > \dim X$ be arbitrary, $\gamma_i = \gamma_i Y$, where $Y^r$ is the $r$-skeleton of $Y$.

Since $\gamma_i \cup \gamma_i = 0$ ($Y$ is an $H'$-space), $\gamma_i \cup \gamma_i = 0$ and $[3]$ there is a map $h_i: Y^r \to S^n$, $n_i = \dim Y_i$, which maps the fundamental class of $S^n$ to some nonzero multiple of $\gamma_i$. Making use of the fact that $Y$ is an $H'$-space we obtain a map $h: Y^r \to V S^{n_i}$ (as in $[3]$) such that $hf$ is a weak $\mathfrak{F}$-equivalence (by the cellular approximation theorem we can assume $f(X) \subset Y^r$). Therefore $X$ is an $H'$-space mod $\mathfrak{F}$ and $f^*: \pi_m(X) \to \pi_m(Y)$ is an $\mathfrak{F}$-monomorphism for $m < r$. Since $r$ was arbitrary the result follows.

(ii) Since the Hurewicz homomorphism for an $H'$-space is an $\mathfrak{F}$-epimorphism in all dimensions, it follows that $h: \pi_m(X) \to H_m(X)$ is an $\mathfrak{F}$-epimorphism for all $m$ and hence that $X$ is an $H'$-space mod $\mathfrak{F}$. Moreover, it is clear that there is a map $h: V S^{n_i} \to Y$ such that $fh$ is a weak $\mathfrak{F}$-equivalence and the result follows.

Remark. In contrast to the Whitehead theorem, the converse of each assertion in Theorems 1 and 2 is false. Counterexamples are given as follows:

1(i). The inclusion map $S^{2n} \to \Omega \Sigma S^{2n}$.

1(ii). The quotient map $S^n \times S^n \to S^n \wedge S^n = S^{2n}$, $n = 3$ or 7.
2(i). The Whitehead product map $S^{m+n-1} \rightarrow S^m \vee S^n$, $m+n$ even, $m, n \geq 2$.

2(ii). The inclusion map $S^m \vee S^n \rightarrow S^m \times S^n$.

BIBLIOGRAPHY


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