A FIXED POINT THEOREM FOR SEMIGROUPS OF MAPPINGS WITH A CONTRACTIVE ITERATE

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Abstract. In a recent paper, Felix E. Browder discussed continuous self-mappings on a metric space, satisfying a functional inequality. Browder gave sufficient conditions such that the successive approximations of any point for such mappings converge to a unique fixed point. In the present paper, Browder's result is extended to a commutative semigroup of mappings and also to single mappings that are not necessarily continuous and satisfy a weaker form of the functional inequality.

1. It is the purpose of this paper to generalize the following result of Felix Browder [2], under the assumption that the set $M$ there is closed.

**Theorem 1.** Let $(X, d)$ be a complete metric space, $M$ a bounded subset of $X$, $T$ a mapping of $M$ into $M$. Suppose there exists a monotone nondecreasing function $\psi(r)$ for $r \geq 0$ with $\psi$ continuous on the right, such that $\psi(r) < r$ for all $r > 0$, while for $x, y \in M$,

$$d(T(x), T(y)) \leq \psi(d(x, y)).$$

Then, for each $x_0 \in M$, $T^n(x_0) \to \xi \in X$, independent of $x_0$, and

$$d(T^n(x_0), \xi) \leq \psi^n(d_0),$$

where $d_0$ is the diameter of $M$, $\psi^n$ is the $n$th iterate of $\psi$, and

$$d_n = \psi^n(d_0) \to 0 \quad \text{as } n \to \infty.$$

It may be remarked that if $M$ is closed, then it follows that $\xi$ is the unique fixed point of $T$ in $M$, and (2) provides an estimate for $d(T^n(x_0), \xi)$. Boyd and Wong [1] have obtained a result similar to Theorem 1, without the estimate (2), where the domain of $T$ is unbounded.

2. Let $(X, d)$ be a complete metric space and $M \subseteq X$. Let $F$ be a commutative semigroup of self-mappings (not necessarily continuous) of $M$. The semigroup $F$ is pointwise contractive in $M$ if for each $x \in M$, there is an $f_x \in F$ such that

$$d(f_x(y), f_x(x)) \leq \psi(d(y, x)).$$
for all \( y \in M \), where \( \psi \) is some real valued function defined on the nonnegative reals.

**Theorem 2.** Let \( M \) be a closed subset of \( X \) and \( F \) a commutative semigroup of self-mappings of \( M \), which is pointwise contractive in \( M \) for some \( \psi : [0, \infty) \rightarrow [0, \infty) \), where \( \psi \) is nondecreasing, continuous on the right and satisfies \( \psi(r) < r \) for all \( r > 0 \). If for some \( x_0 \in M \),

\[
\sup \{ d(f(x_0), x_0) : f \in M \} < \infty,
\]

then, there exists a unique \( \xi \in M \) such that \( f(\xi) = \xi \) for each \( f \in F \). Moreover, there is a sequence \( \{ g_n \} \subseteq F \) with \( g_n(x) \rightarrow \xi \) for each \( x \in M \).

**Proof.** Let \( d_0 = \sup \{ d(f(x_0), x_0) : f \in F \} \). Then, \( \psi^n(d_0) \) is a nonincreasing sequence of nonnegative reals, and therefore, \( \psi^n(d_0) \rightarrow \infty \). If \( r > 0 \), then \( \psi(\lim_n \psi^n(d_0)) < r \), that is, \( r = \lim_n \psi^n(d_0) \). We conclude, therefore, that \( \lim_n \psi^n(d_0) = 0 \).

Set \( f_0 = f_{x_0} \) and inductively \( f_n = f_{x_n} \) where \( x_{n+1} = f_n(x_n) \). Then, for a fixed integer \( k \geq 0 \),

\[
\sup_{n \geq k} d(x_{n+1}, x_{k+1}) = \sup_{n \geq k} d(f_n \cdot f_{n-1} \cdots f_k(x_k), f_k(x_k)).
\]

Set \( h_n = f_n \cdot f_{n-1} \cdots f_{k+1} \). It follows that

\[
\sup_{n \geq k} d(x_{n+1}, x_{k+1}) = \sup_{n \geq k} d(h_n(x_k), f_k(x_k)) \leq \sup_{n \geq k} \psi(d(h_n(x_k), x_k)) \leq \sup_{n \geq k} \psi^{k+1}(d(h_n(x_0), x_0)) \leq \psi^{k+1}(d_0) \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty.
\]

The sequence \( \{ x_n \} \) is therefore, Cauchy. Let \( x_n \rightarrow \xi \in M \). Then, by hypothesis, there is a \( f_\xi \in F \) such that

\[
d(f_\xi(x_n), f_\xi(\xi)) \leq \psi(d(x_n, \xi)) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

Thus, \( f_\xi(x_n) \rightarrow f_\xi(\xi) \), and therefore \( d(f_\xi(\xi), \xi) = \lim_n d(f_\xi(x_n), x_n) \). However,

\[
d(f_\xi(x_n), x_n) \leq \psi(d(f_\xi(x_{n-1}), x_{n-1})) \leq \psi^n(d(f_\xi(x_0), x_0)) \leq \psi^n(d_0) \rightarrow 0
\]

as \( n \rightarrow \infty \). Hence \( f_\xi(\xi) = \xi \). It follows from (3) that \( \xi \) is the unique fixed point of \( f_\xi \). Furthermore, by the commutativity of \( F \), we have, for any \( f \in F \),

\[
f(\xi) = f(f_\xi(\xi)) = f_\xi(f(\xi)),
\]

and therefore \( f(\xi) = \xi \).
Finally, for each nonnegative integer $n$, set $g_n = f_n \cdot f_{n-1} \cdot \cdots \cdot f_0$. Then $g_n \in F$. We show that $g_n(x) \to \xi$ for each $x \in M$. If $d(x) = d(x, x_0)$, then it follows that $\psi^n(d(x)) \to 0$. We have

$$d(g_n(x), \xi) \leq d(g_n(x), x_{n+1}) + d(x_{n+1}, \xi).$$

Since $x_n \to \xi$, it suffices to show that $d(g_n(x), x_{n+1}) \to 0$. However,

$$d(g_n(x), x_{n+1}) = d(f_n(g_{n-1}(x)), f_n(x_n)),$$

$$\leq \psi(g_{n-1}(x), x_n),$$

$$\leq \psi^{n+1}(d(x, x_0)) = \psi^{n+1}(d(x)) \to 0 \text{ as } n \to \infty.$$

Thus $d(g_n(x), x_{n+1}) \to 0$. This completes the proof.

If $M$ is a bounded subset of $X$, then since (4) holds for each $x_0 \in M$, we have

**Corollary 1.** Let $M$ be a closed bounded subset of $X$, and $F$ a commutative semigroup of self-mappings of $M$ which is pointwise contractive in $M$ for some $\psi: [0, \infty) \to [0, \infty)$, where $\psi$ is nondecreasing, continuous on the right and satisfies $\psi(r) < r$ for $r > 0$. Then there exist a sequence $\{g_n\} \subseteq F$ and a unique $\xi \in M$ such that $f(x) = \xi$ for all $f \in F$, and $g_n(x) \to \xi$ for each $x \in M$.

**Corollary 2.** Let $M$ be a closed bounded subset of $X$, and $f$ a self-mapping of $M$. If $f$ satisfies the condition: for each $x \in M$, there exists an integer $n(x) \geq 1$ such that, for all $y \in M$,

$$d(f^{n(x)}(y), f^{n(x)}(x)) \leq \psi(d(y, x)),$$

where $\psi: [0, \infty) \to [0, \infty)$ is nondecreasing, continuous on the right and satisfies $\psi(r) < r$ for $r > 0$, then there is a unique $\xi \in M$ such that $f(x) = \xi$ for each $x \in M$ and $f(\xi) = \xi$.

**Proof.** Since $F = \{f^n : n \geq 0\}$ is a commutative semigroup which is pointwise contractive in $M$, the existence and uniqueness of $\xi \in M$ follow by Corollary 1. We show that $f^n(x) \to \xi$ for each $x \in M$. Let $d_0$ be the diameter of the set $M$. Then if $n$ is sufficiently large, we have $n = r \cdot n(\xi) + s$, with $r > 0$ and $0 \leq s < n(\xi)$, and, therefore,

$$d(f^n(x), \xi) = d(f^{r \cdot n(\xi) + s}(x), f^{n(\xi)}(\xi)) \leq \psi^{r}(d(f^s(x), \xi)) \leq \psi(r).$$

Since $\psi(r) \to 0$, and $r \to \infty$ as $n \to \infty$, it follows therefore that $f^n(x) \to \xi$.

**Remark 1.** It should be noted that inequality (6) provides the analogue to Browder’s estimate (2).

**Remark 2.** For bounded metric spaces, Corollary 2 is an extension of a result of the first author [4] and also of a recent result of L. F. Guseman [2]. Note that Theorem 2 and its corollaries provide
generalizations of Theorem 1 under the assumption that the set $M$ is closed.

References


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