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A FIXED POINT THEOREM FOR SEMIGROUPS OF MAPPINGS WITH A CONTRACTIVE ITERATE

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ABSTRACT. In a recent paper, Felix E. Browder discussed continuous self-mappings on a metric space, satisfying a functional inequality. Browder gave sufficient conditions such that the successive approximations of any point for such mappings converge to a unique fixed point. In the present paper, Browder's result is extended to a commutative semigroup of mappings and also to single mappings that are not necessarily continuous and satisfy a weaker form of the functional inequality.

1. It is the purpose of this paper to generalize the following result of Felix Browder [2], under the assumption that the set M there is closed.

THEOREM 1. *Let (X, d) be a complete metric space, M a bounded subset of X, T a mapping of M into M. Suppose there exists a monotone nondecreasing function $\psi(r)$ for $r \geqq 0$ with ψ continuous on the right, such that $\psi(r) < r$ for all $r > 0$, while for $x, y \in M$,*

$$d(T(x), T(y)) \leqq \psi(d(x, y)). \tag{1}$$

Then, for each $x_0 \in M$, $T^n(x_0) \rightarrow \xi \in X$, independent of x_0, and

$$d(T^n(x_0), \xi) \leqq \psi^n(d_0), \tag{2}$$

where d_0 is the diameter of M, ψ^n is the nth iterate of ψ, and

$$d_n = \psi^n(d_0) \rightarrow 0 \quad as\ n \rightarrow \infty.$$

It may be remarked that if M is closed, then it follows that ξ is the unique fixed point of T in M, and (2) provides an estimate for $d(T^n(x_0), \xi)$. Boyd and Wong [1] have obtained a result similar to Theorem 1, without the estimate (2), where the domain of T is unbounded.

2. Let (X, d) be a complete metric space and $M \subseteq X$. Let F be a commutative semigroup of self-mappings (not necessarily continuous) of M. The semigroup F is pointwise contractive in M if for each $x \in M$, there is an $f_x \in F$ such that

$$d(f_x(y), f_x(x)) \leqq \psi(d(y, x)), \tag{3}$$

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for all $y \in M$, where ψ is some real valued function defined on the nonnegative reals.

THEOREM 2. *Let M be a closed subset of X and F a commutative semigroup of self-mappings of M, which is pointwise contractive in M for some $\psi: [0, \infty) \to [0, \infty)$, where ψ is nondecreasing, continuous on the right and satisfies $\psi(r) < r$ for all $r > 0$. If for some $x_0 \in M$,*

$$\sup\{d(f(x_0), x_0): f \in M\} < \infty, \tag{4}$$

then, there exists a unique $\xi \in M$ such that $f(\xi) = \xi$ for each $f \in F$. Moreover, there is a sequence $\{g_n\} \subseteq F$ with $g_n(x) \to \xi$ for each $x \in M$.

PROOF. Let $d_0 = \sup\{d(f(x_0), x_0): f \in F\}$. Then, $\psi^n(d_0)$ is a nonincreasing sequence of nonnegative reals, and therefore, $\psi^n(d_0) \to r \geqq 0$. If $r > 0$, then $\psi(\lim_n \psi^n(d_0)) < r$, that is, $r = \lim_n \psi^n(d_0) < r$. We conclude, therefore, that $\lim_n \psi^n(d_0) = 0$.

Set $f_0 = f_{x_0}$ and inductively $f_n = f_{x_n}$ where $x_{n+1} = f_n(x_n)$. Then, for a fixed integer $k \geqq 0$,

$$\sup_{n \geqq k} d(x_{n+1}, x_{k+1}) = \sup_{n \geqq k} d(f_n \cdot f_{n-1} \cdots f_k(x_k), f_k(x_k)).$$

Set $h_n = f_n \cdot f_{n-1} \cdots f_{k+1}$. It follows that

$$\begin{aligned} \sup_{n \geqq k} d(x_{n+1}, x_{k+1}) &= \sup_{n \geqq k} d(f_k(h_n(x_k)), f_k(x_k)), \\ &\leqq \sup_{n \geqq k} \psi(d(h_n(x_k), x_k)), \\ &\leqq \sup_{n \geqq k} \psi^{k+1}(d(h_n(x_0), x_0)), \\ &\leqq \psi^{k+1}(d_0) \to 0 \quad \text{as } k \to \infty. \end{aligned}$$

The sequence $\{x_n\}$ is therefore, Cauchy. Let $x_n \to \xi \in M$. Then, by hypothesis, there is an $f_\xi \in F$ such that

$$d(f_\xi(x_n), f_\xi(\xi)) \leqq \psi(d(x_n, \xi)) \to 0 \quad \text{as } n \to \infty.$$

Thus, $f_\xi(x_n) \to f_\xi(\xi)$, and therefore $d(f_\xi(\xi), \xi) = \lim_n d(f_\xi(x_n), x_n)$. However,

$$d(f_\xi(x_n), x_n) \leqq \psi(d(f_\xi(x_{n-1}), x_{n-1})) \leqq \psi^n(d(f_\xi(x_0), x_0)) \leqq \psi^n(d_0) \to 0$$

as $n \to \infty$. Hence $f_\xi(\xi) = \xi$. It follows from (3) that ξ is the unique fixed point of f_ξ. Furthermore, by the commutativity of F, we have, for any $f \in F$,

$$f(\xi) = f(f_\xi(\xi)) = f_\xi(f(\xi)),$$

and therefore $f(\xi) = \xi$.

Finally, for each nonnegative integer n, set $g_n = f_n \cdot f_{n-1} \cdots f_0$. Then $g_n \in F$. We show that $g_n(x) \to \xi$ for each $x \in M$. If $d(x) = d(x, x_0)$, then it follows that $\psi^n(d(x)) \to 0$. We have

$$d(g_n(x), \xi) \leqq d(g_n(x), x_{n+1}) + d(x_{n+1}, \xi).$$

Since $x_n \to \xi$, it suffices to show that $d(g_n(x), x_{n+1}) \to 0$. However,

$$\begin{aligned} d(g_n(x), x_{n+1}) &= d(f_n(g_{n-1}(x)), f_n(x_n)), \\ &\leqq \psi(g_{n-1}(x), x_n), \\ &\leqq \psi^{n+1}(d(x, x_0)) = \psi^{n+1}(d(x)) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Thus $d(g_n(x), x_{n+1}) \to 0$. This completes the proof.

If M is a bounded subset of X, then since (4) holds for each $x_0 \in M$, we have

Corollary 1. *Let M be a closed bounded subset of X, and F a commutative semigroup of self-mappings of M which is pointwise contractive in M for some $\psi: [0, \infty) \to [0, \infty)$, where ψ is nondecreasing, continuous on the right and satisfies $\psi(r) < r$ for $r > 0$. Then there exist a sequence $\{g_n\} \subseteq F$ and a unique $\xi \in M$ such that $f(\xi) = \xi$ for all $f \in F$, and $g_n(x) \to \xi$ for each $x \in M$.*

Corollary 2. *Let M be a closed bounded subset of X, and f a self-mapping of M. If f satisfies the condition: for each $x \in M$, there exists an integer $n(x) \geqq 1$ such that, for all $y \in M$,*

$$d(f^{n(x)}(y), f^{n(x)}(x)) \leqq \psi(d(y, x)), \tag{5}$$

where $\psi: [0, \infty) \to [0, \infty)$ is nondecreasing, continuous on the right and satisfies $\psi(r) < r$ for $r > 0$, then there is a unique $\xi \in M$ such that $f^n(x) \to \xi$ for each $x \in M$ and $f(\xi) = \xi$.

Proof. Since $F = \{f^n : n \geqq 0\}$ is a commutative semigroup which is pointwise contractive in M, the existence and uniqueness of $\xi \in M$ follow by Corollary 1. We show that $f^n(x) \to \xi$ for each $x \in M$. Let d_0 be the diameter of the set M. Then if n is sufficiently large, we have $n = r \cdot n(\xi) + s$, with $r > 0$ and $0 \leqq s < n(\xi)$, and, therefore,

$$d(f^n(x), \xi) = d(f^{r \cdot n(\xi) + s}(x), f^{n(\xi)}(\xi)) \leqq \psi^r(d(f^s(x), \xi)) \leqq \psi^r(d_0). \tag{6}$$

Since $\psi^n(d_0) \to 0$, and $r \to \infty$ as $n \to \infty$, it follows therefore that $f^n(x) \to \xi$.

Remark 1. It should be noted that inequality (6) provides the analogue to Browder's estimate (2).

Remark 2. For bounded metric spaces, Corollary 2 is an extension of a result of the first author [4] and also of a recent result of L. F. Guseman [2]. Note that Theorem 2 and its corollaries provide generalizations of Theorem 1 under the assumption that the set M is closed.

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