EXISTENCE AND CONTINUOUS DEPENDENCE FOR A
CLASS OF NONLINEAR NEUTRAL-DIFFERENTIAL EQUATIONS

L. J. GRIMM

Abstract. This paper presents existence, uniqueness, and continuous dependence theorems for solutions of initial-value problems for neutral-differential equations of the form

\begin{align*}
x'(t) &= f(t, x(t), x(g(t, x)), x'(h(t, x))), \\
x(0) &= x_0,
\end{align*}

where \( f, g, \) and \( h \) are continuous functions with \( g(0, x_0) = h(0, x_0) = 0 \). The existence of a continuous solution of the functional equation \( z(t) = f(t, z(h(t))) \) is proved as a corollary.

1. Introduction. In this paper we consider the initial-value problem (IVP) for the functional-differential equation of neutral type

\begin{equation}
x'(t) = f(t, x(t), x(g(t, x(t))), x'(h(t, x(t)))),
\end{equation}

with the initial condition

\begin{align*}
(2a) & \quad x(0) = x_0. \\
(2b) & \quad x'(0) = z_0.
\end{align*}

Here \( f(t, x, y, z), g(t, x) \) and \( h(t, x) \) are continuous functions with \( g(0, x_0) = h(0, x_0) = 0 \). We assume further that the algebraic equation \( z = f(0, x_0, x_0, z) \) has a real root \( z_0 \), and we require that

Existence and uniqueness theorems for IVP's for equation (1) have been proved by R. D. Driver [1] for the case where \( h(t, x) < t \), and recently by J. K. Hale and M. A. Cruz [3] provided that \( f \) is linear in the argument \( x'(h(t, x)) \). We prove an existence theorem without these hypotheses, and a uniqueness theorem in case \( h \) is independent of \( x \). Hale and Cruz [3] have also obtained continuity theorems for the quasilinear case mentioned above, while Driver [2] has proved a continuity theorem for IVP's for equations of the form (1) in case \( g \) and \( h \) are both independent of \( x \), and \( h(t) < t \) for all \( t \). We obtain here a continuous dependence result for the IVP (1)−(2a)−(2b) provided that the function \( h \) is independent of \( x \). Finally we obtain a result on existence of continuous solutions of certain nonlinear functional equations as a corollary of our existence and uniqueness theorems.

Received by the editors November 4, 1970.

AMS 1970 subject classifications. Primary 34K05; Secondary 34K05.

Key words and phrases. Neutral-differential equations, functional equations, continuous dependence, existence theory.

1 Research supported by National Science Foundation Grant GP 20194.

Copyright © 1971, American Mathematical Society

467
2. Existence. Let $\alpha > 0$, and let $J = [-\alpha, \alpha]$. We shall make the following assumptions concerning the IVP (1)–(2a)–(2b):

(i) $f(t, x, y, z)$ is continuous in some region in $\mathbb{R}^4$ containing

$$P = \{ (t, x, y, z): |t| \leq \alpha, |x - x_0| \leq \beta, |y - x_0| \leq \beta, |z| \leq M \}$$

where $\alpha$, $\beta$ and $M > |z_0|$ are positive constants. We assume that $\alpha \leq \beta/M$ and that $\sup_{(t,x,y,z) \in P} |f(t, x, y, z)| < M$.

(ii) $g(t, x)$ and $h(t, x)$ are continuous in the projection $\tilde{R}$ of $P$ in the $(t, x)$ space; $g$ and $h$ both map $\tilde{R}$ into $\mathbb{R}$, with $g(0, x_0) = h(0, x_0) = 0$, and $h(t, x)$ satisfies the Lipschitz conditions

$$|h(t_1, x_1) - h(t_2, x_2)| \leq k_1 |t_1 - t_2| + k_2 |x_1 - x_2|$$

for all $(t_1, x_1), (t_2, x_2) \in \tilde{R}$, where $k_1$ and $k_2$ are nonnegative constants with $k_1 + k_2 M \leq 1$.

(iii) The function $f(t, x, y, z)$ satisfies the Lipschitz condition

$$|f(t_1, x, y_1, z_1) - f(t_2, x, y_2, z_2)| \leq L_z |z_1 - z_2|$$

for all $(t, x, y_1, z_1), (t, x, y_2, z_2) \in P$, where $L_z < 1$.

We shall prove the following theorem:

**Theorem 1.** Under the hypotheses (i)–(iii), the IVP (1)–(2a)–(2b) has at least one solution which is continuously differentiable on $J$.

**Proof.** Let $X$ be the Banach space of continuous functions on $J$ with uniform norm. Let

$$S = \{ z \in X: z(0) = x_0, \|z\| \leq M \}.$$ 

Define the mapping $T: S \to S$ as follows: for $z \in S$, let

$$Tz(t) = f(t, I(z, t), I(z, g(t, I(z, t))), z(h(t, I(z, t)))),$$

where

$$I(z, t) = x_0 + \int_0^t z(s)ds.$$ 

It is easy to verify that $T$ is a continuous map of $S$ into $S$. By continuity of $f$, if $z \in S$ and $t \in J$, for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if $t_1$ and $t_2 \in J$, and $|t_1 - t_2| < \delta(\varepsilon)$, then

$$|f(t_1, I(z, t_1), I(z, g(t_1, I(z, t_1))), z(h(t, I(z, t)))) - f(t_2, I(z, t_2), I(z, g(t_2, I(z, t_2))), z(h(t, I(z, t))))| < \varepsilon.$$
\[ S_\varepsilon = \{ z \in S : | z(t_1) - z(t_2) | \leq \varepsilon/(1 - L_\varepsilon) \]

for all \( t_1, t_2 \in J, | t_1 - t_2 | \leq \delta(\varepsilon) \).

If \( z \in S_\varepsilon \), and if \( t_1, t_2 \in J \) with \( | t_1 - t_2 | \leq \delta(\varepsilon) \), then

\[ | Tz(t_1) - Tz(t_2) | \leq \varepsilon + \varepsilon L_\varepsilon/(1 - L_\varepsilon) = \varepsilon/(1 - L_\varepsilon). \]

Thus \( TS_\varepsilon \subseteq S_\varepsilon \). We note that \( S_\varepsilon \) is closed, bounded and convex. Let

\[ S_0 = \bigcap_{\varepsilon > 0} S_\varepsilon. \]

\( S_0 \) is nonempty, closed, bounded and convex, and by the Ascoli-Arzela theorem, \( S_0 \) is compact. Since \( TS_\varepsilon \subseteq S_\varepsilon \) for all \( \varepsilon > 0 \), \( TS_0 \subseteq S_0 \). Hence by the Schauder theorem, \( T \) has at least one fixed point \( z(t) \). Integration yields the required solution of (1)-(2a)-(2b).

3. **Uniqueness.** In case \( h(t, x) \) is independent of \( x \), we obtain the following uniqueness result:

**Theorem 2.** In addition to the hypotheses of Theorem 1, suppose that:

- \( h(t, x) = h(t) \) is independent of \( x \).
- \( f \) and \( g \) satisfy the Lipschitz conditions

\[ | f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2) | \leq L \left( | x_1 - x_2 | + | y_1 - y_2 | \right) + L_\varepsilon | z_1 - z_2 | \]

with \( L_\varepsilon < 1 \);

\[ | g(t, x_1) - g(t, x_2) | \leq L_\gamma | x_1 - x_2 |, \]

uniformly in their domains.

Then there exists \( \gamma_0, 0 < \gamma_0 \leq \alpha, \) such that there is a unique continuously differentiable solution of the IVP (2)-(3a)-(3b) on \([ -\gamma_0, \gamma_0 ]\).

**Proof.** Under the hypotheses of the theorem, if \( z \in S, 0 < \gamma \leq \alpha, \) and \( t \in [ -\gamma, \gamma ] \),

\[ | Tz_1(t) - Tz_2(t) | \leq L \left\{ I(z_1, t) - I(z_2, t) \right\} \]

\[ + L_\varepsilon \left| z_1(h(t)) - z_2(h(t)) \right| \]

\[ \leq L_\gamma | z_1 - z_2 | + L_\gamma | z_1 - z_2 | \]

\[ + LL_\varepsilon M_\gamma | z_1 - z_2 | + L_\varepsilon | z_1 - z_2 | \]

\[ = \left[ L_\gamma (2 + ML_\varepsilon) + L_\varepsilon \right] | z_1 - z_2 |. \]

Hence if \( \gamma \) is sufficiently small, the mapping \( T \) is a contraction, and the statement of the theorem follows by integration.
Remark. A uniqueness theorem will follow also from the theorem in the next section.

4. Continuous dependence. For \( i=1, 2 \), consider the IVP's

\[
\begin{align*}
(1.i) & \quad x'_i(t) = f_i(t, x_i(t), x_i(g_i(t, x_i(t))), x'_i(h_i(t))), \\
(2.ia) & \quad x_i(0) = x_{i0}, \\
(2.ib) & \quad x'_i(0) = z_{i0},
\end{align*}
\]

under hypotheses analogous to (i)-(v):

\( (H1) \) For \( i=1, 2 \), \( f_i(t, x, y, z) \) is continuous in some domain \( D \subset R^4 \) containing both of the sets

\[ P_i = \{ (t, x, y, z) : |t| \leq a, |x - x_{i0}| \leq b, |y - x_{i0}| \leq b, |z| \leq M \}, \]

where \( x_{i0} \) are constants, \( a, b \), and \( M > |z_{i0}| \) are constants with \( \sup_{(t, x, y, z) \in D} |f_i(t, x, y, z)| < M \), and \( z_{i0} \) is a real root of the equation \( z = f_i(t, x_{i0}, x_{i0}, z) \).

\( (H2) \) For \( i=1, 2 \), \( g_i(t, x) \) is continuous in the projection of \( D \) in the \( (t, x) \) plane, and \( h_i(t) \) is continuous on \( [-a, a] \), with \( |g_i(t, x)| \leq |t| ; |h_i(t)| \leq |t| \).

\( (H3) \) The functions \( f_i \) and \( g_i \) satisfy the conditions satisfied by \( f \) and \( g \) respectively in §3.

Theorem 3. Let \( (H1)-(H3) \) be satisfied, let \( \alpha = \min (a, b/M) \) and suppose that for \( i=1, 2 \), \( x_i(t) \) is a continuously differentiable function which satisfies (1.i) - (2.ia) - (2.ib), with

\[ |x_{i0} - x_{20}| = \varepsilon_0 < \alpha M, \]

and there exist nonnegative constants \( \varepsilon_f, \varepsilon_g, \varepsilon_h \) such that

\[ |f_1(t, x, y, z) - f_2(t, x, y, z)| \leq \varepsilon_f, \]
\[ |g_1(t, x) - g_2(t, x)| \leq \varepsilon_g, \]
\[ |h_1(t) - h_2(t)| \leq \varepsilon_h \]

in their respective domains. Then if \( \varepsilon_h \) is sufficiently small, for all \( t \in [-\alpha, \alpha] \),

\[ |x_1(t) - x_2(t)| \leq \varepsilon_0 + C_{\varepsilon, x_1} \left[ \exp \left( \frac{(2 + ML^2) t}{1 - L} \right) - 1 \right] \]

where

\[ C_{\varepsilon, x_1} = \frac{\varepsilon_f + (2 + ML^2) \varepsilon_0 + ML^2 \varepsilon_0 + L \varepsilon_{x_1, h}}{L(2 + ML^2)} \]

and for each fixed solution \( x_1(t) \), the quantity \( \varepsilon_{x_1, h} \) tends to zero as \( \varepsilon_h \to 0 \).
Proof. Let $\eta > 0$. By continuity of $x_i'(t)$, there exists $\delta > 0$ such that if $t, \tau \in [0, \alpha]$ and $|t - \tau| < \delta$, then $|x_i'(t) - x_i'(\tau)| < \eta$. We suppose that $\epsilon_\theta < \delta$. Set $z_i(t) = x_i'(t)$, $i = 1, 2$. The functions $z_i$ satisfy the equations

$$z_i(t) = f_i(t, x_10 + \int_0^t z_i(s) \, ds),$$

(4.1)

$$x_10 + \int_0^t g_i(t, x_10 + \int_0^t z_i(s) \, ds, z_i(t), z_i(h_2(t))).$$

Using the Lipschitz continuity of $f_i$, and the definitions of the quantities $\epsilon_\theta, \epsilon_f, \epsilon_g$ and $\eta$, we obtain from (4.1) and (4.2) the estimate

$$|z_1(t) - z_2(t)| \leq \eta \left\{ \epsilon_\theta + \left| \int_0^t |z_1(s) - z_2(s)| \, ds \right| \right\}$$

$$+ L \left\{ \epsilon_\theta + \left| \int_0^t g_i(t, x_10 + \int_0^t z_i(s) \, ds) \right| |z_3(s) - z_4(s)| \, ds \right\}$$

$$+ \left| \int_2 g_i(t, x_10 + \int_0^t z_i(s) \, ds) \right| |z_1(s)| \, ds$$

$$+ \left| \int_3 g_i(t, x_10 + \int_0^t z_i(s) \, ds) \right| |z_1(s)| \, ds$$

$$+ L \left| z_1(h_2(t)) - z_2(h_2(t)) \right| + L \eta.$$
Let $K = \epsilon_\tau + (2 + ML_\theta)L_\phi + ML_\eta + L_\eta\eta$, and

$$R(t) = \max_{1 \leq |t| \leq |t|} |z_1(s) - z_2(s)|.$$ 

Then, on $[0, \alpha]$ we have

$$R(t) \leq K + (2 + ML_\theta)L \int_0^t R(s)ds + L_2R(h_2(t)),$$

and since $R$ is an even function, is nondecreasing, and $|h_2(t)| \leq |t|$, 

$$R(t) \leq \frac{K}{1 - L_2} + \frac{(2 + ML_\theta)L}{1 - L_2} \int_0^t R(s)ds.$$ 

By the Gronwall inequality

$$R(t) \leq \frac{K}{1 - L_2} \exp\left(\frac{(2 + ML_\theta)Lt}{1 - L_2}\right)$$

and integration leads to

$$|z_1(t) - z_2(t)| \leq \epsilon_0 + \int_0^t R(s)ds$$

$$\leq \epsilon_0 + \frac{K}{(2 + ML_\theta)L} \left[\exp\left(\frac{(2 + ML_\theta)Lt}{1 - L_2}\right) - 1\right],$$

and setting $C_{z_1,z_2} = K/((2 + ML_\theta)L)$, we obtain (3) on $[0, \alpha]$. Since $R$ is an even function, the estimate (5) holds on $[-\alpha, 0]$ if $t$ is replaced by $-t$. Thus analogously the estimate (3) holds also on $[-\alpha, 0]$ and the proof is complete.

5. **Nonlinear functional equations.** As a corollary to our existence and uniqueness results, we note that if $f(t, x, y, z)$ is independent of $x$ and $y$, and $h(t, x)$ is independent of $x$, the problem (1)–(2b) has the form of the functional equation

(5) $z(t) = f(t, z(h(t)))$,

(6) $z(0) = z_0$,

where $z_0$ is a root of $z = f(0, z)$. Theorems 1 and 2 then yield at once:

**Theorem 4.** Let $f(t, z)$ be continuous in some region in $\mathbb{R}^2$ containing $P_1 = \{t: |t| \leq \alpha, |z| \leq M\}$, where $\alpha$ and $M$ are positive constants such that $\sup_{(t, z) \in P_1} |f(t, z)| < M$, and $M > |z_0|$ where $z_0$ is a real root of $z = f(0, z)$. Let $f$ satisfy the Lipschitz condition $|f(t, z_1) - f(t, z_2)| \leq L_2|z_1 - z_2|$ for all $(t, z_1), (t, z_2) \in P_1$, with $L_2 < 1$. Let $h(t)$ be continuous for $|t| \leq \alpha$, $h(0) = 0$, and $|h(t_1) - h(t_2)| \leq |t_1 - t_2|$ for $t_1, t_2 \in [-\alpha, \alpha]$. 
The problem (5)–(6) has at least one continuous solution on \([-\alpha, \alpha]\), and this is the unique continuous solution on this interval if \(\alpha\) is sufficiently small.

References


University of Missouri-Rolla, Rolla, Missouri 65401