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CONDITIONS FOR CONTINUITY OF CERTAIN OPEN MONOTONE FUNCTIONS

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ABSTRACT. In this paper continuity of certain open monotone functions is obtained by assuming for the domain and/or range various combinations of the properties of a metric continuum, regular metric continuum, semilocal connectedness, and hereditary local connectedness. An open monotone connected function from a hereditarily locally connected separable metric continuum onto a separable metric continuum is continuous. If the domain is a regular separable metric continuum, an upper semicontinuous decomposition and resulting monotone-light factorization yield continuity of an open monotone function with closed point inverses.

By a continuum is meant a compact connected space. A function f is monotone if point inverses are connected. If f is a function from X onto Y, the component decomposition X' of X induced by f is the collection of all components of sets of the form $f^{-1}(y)$, where y varies over Y. A function is connected if it takes connected sets onto connected sets. A continuum is regular provided every point has arbitrarily small open neighborhoods with finite boundaries [**4**]. It should be noted that this is not the same as a regular topological space as usually defined.

THEOREM 1. *If f is an open monotone connected function from the 1st countable space X onto the 1st countable semilocally connected space Y, then f is continuous.*

PROOF. If f is not continuous there exists an open set U in Y such that $f^{-1}(U)$ is not open in X. Hence there is a point $x \in f^{-1}(U)$ and a sequence $\{x_n\}$ of distinct points in $X - f^{-1}(U)$ such that $x_n \to x$. Since Y is semilocally connected there exists an open set $V \subset U$ such that $Y - V$ has only a finite number of components. Since $f(x_n) \notin V$ for all n it follows that some component C of $Y - V$ contains $f(x_n)$ for infinitely many n. By Theorem 2 of [**1**], $f^{-1}(C)$ is connected, and x is a limit point of $f^{-1}(C)$. Hence $f^{-1}(C) \cup \{x\}$ is connected but

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$f(f^{-1}(C)\cup\{x\})=C\cup\{f(x)\}$ is not connected since C is closed and $f(x)\notin C$. This contradicts f being connected. Thus f is continuous.

COROLLARY 1. *If f is an open monotone connected function from the 1st countable space X onto the hereditarily locally connected separable metric continuum Y, then f is continuous.*

PROOF. By Theorem 13.21, p. 20 of [**4**], Y is semilocally connected.

THEOREM 2. *If f is an open monotone connected function from X onto Y, where X and Y are separable metric continua and X is hereditarily locally connected, then f is continuous.*

PROOF. By Theorem 4 of [**1**], Y is hereditarily locally connected. Hence by Corollary 1, f is continuous.

THEOREM 3. *Let f be a function from the T_1 space X onto the semi-locally connected T_1 space Y with the following properties:*

(a) *f is finite-to-1 onto Y,*

(b) *the inverse image $f^{-1}(H)$ of a closed set H in Y has closed components, and*

(c) *if C is a connected subset of Y, then every component of $f^{-1}(C)$ maps onto all of C.*

Then f is continuous.

PROOF. If f is not continuous at a point p in X, then there is an open set V containing $f(p)$ such that if U is any open set containing p, $f(U)$ is not a subset of V. Since Y is semilocally connected, there is an open set $W\subset V$ and containing $f(p)$ such that $Y-W$ has a finite number of components, $C_1, \cdots, C_n$. Now $Y-W$ closed implies that the C_i are closed, and f finite-to-1 implies that $f^{-1}(C_i)$ has a finite number of components K_{ij}, since each component maps onto all of C_i. The point p is a limit point of at least one component of $f^{-1}(C_i)$ for some i. For if for every i, p is not a limit point of any component of $f^{-1}(C_i)$, then there is an open set U_{ij} containing p and disjoint from K_{ij} for all i and j. If U denotes the intersection of all the U_{ij}, then U is an open set containing p such that $f(U)\cap(Y-W)=\emptyset$. Thus $f(U)\subset W\subset V$. This contradicts the hypothesis that $f(U)$ is not contained in V for any open set U. Thus p is a limit point of some component of some $f^{-1}(C_i)$. But p is not in $f^{-1}(C_i)$ contradicting the hypothesis that $f^{-1}(C_i)$ has closed components. Therefore f is continuous.

THEOREM 4. *If X is a regular separable metric continuum and G is a decomposition of X into disjoint continua, then G is upper semicontinuous.*

PROOF. Let $\{D_n\}$ be a sequence of sets from G and $L = \lim\sup D_n$. If $\{D_n\}$ is not a null sequence, then there is a positive number δ and a positive integer N such that $n > N$ implies that $\operatorname{diam}(D_n) > \delta$. Since X is compact, $L \neq \emptyset$. Let $p \in L$. Since X is regular at p there is an open set R with diameter less than δ such that $F(R)$, the boundary of R, is finite. Since $\operatorname{diam}(D_n) > \delta$ for $n > N$, $D_n - R$ and $R - D_n$ are nonempty. If $F(R) \cap D_n = \emptyset$, then $D_n - R$ and $R - D_n$ form a separation for the connected set D_n. Thus $F(R) \cap D_n \neq \emptyset$ for all $n > N$. Since the D_n are disjoint, $F(R)$ must contain infinitely many points. But $F(R)$ is finite. Hence $\{D_n\}$ must be a null sequence and L is a singleton. Thus L is contained in a single element of G and hence G is upper semicontinuous [4, p. 122].

THEOREM 5. *If f is a function from X onto Y, where X and Y are separable metric continua, X is regular, and components of sets $f^{-1}(y)$, $y \in Y$, are closed, then f can be factored into the composite $f = f_2 f_1$, where f_1 from X onto X' is monotone and continuous and f_2 from X' onto Y is light.*

PROOF. Since X is compact, X' is a collection of disjoint continua filling up X. Thus, by Theorem 4, X' is upper semicontinuous. Define f_1 from X onto X' by $f_1(x) = C$ if and only if $x \in C$, and define f_2 from X' onto Y by $f_2(C) = y$ if and only if C is a component of $f^{-1}(y)$. That f_1 and f_2 have the desired properties follows as in the proof of Theorem 5 of [1].

THEOREM 6. *If f is an open monotone function from X onto Y, where X and Y are separable metric continua, X is regular, and $f^{-1}(y)$ is closed for all $y \in Y$, then f is continuous.*

PROOF. Let $f = f_2 f_1$ be the factorization given in Theorem 5. Since f is monotone, $f^{-1}(y)$ is a continuum and thus has only one component. Therefore f_2 is a one-to-one function from X' onto Y. The function f_2 is also open since if A is an open set in X', then A^* (the point set union of elements in A) is open in X. Thus $f(A^*) = f_2(f_1(A^*)) = f_2(A)$ is open in Y since f is an open function. Hence f_2 is an open function. Since f_2 is an open one-to-one function, f_2^{-1} is a continuous function from Y onto X'. But X' is a compact metric space. Therefore $(f_2^{-1})^{-1} = f_2$ is continuous [4, p. 25]. Hence f is the composite of two continuous functions and thus is continuous.

COROLLARY 2. *If f is an open monotone function from X onto Y, where X and Y are separable metric continua, X is regular and f is either a connected, connectivity, or peripherally continuous function, then f is continuous.*

PROOF. For connected, connectivity, and peripherally continuous functions, point inverses have closed components [2], [3]. Thus if f is monotone, $f^{-1}(y)$ is closed for each $y \in Y$.

THEOREM 7. *If f is an open monotone connected function from the space X onto the semilocally connected* 1*st countable space Y, and if X' is upper semicontinuous, then f is continuous.*

PROOF. Just as in the proof of Theorem 6, f can be factored into the composite $f = f_2 f_1$, where f_1 is monotone and continuous from X onto X' and f_2 is one-to-one and open from X' onto Y. Just as in the proof of Theorem 5 of [1], f_2 is a connected function since f is connected. By Theorem 2 of [1], f_2^{-1} is a connected function. Therefore f_2 is a biconnected function and by Theorem 3.7 of [3] is continuous. Thus, f is continuous.

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