RINGS GENERATED BY THE INNER-AUTOMORPHISMS OF NONABELIAN GROUPS

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Abstract. The endomorphisms of an abelian group form a ring. The characterization of groups in which endomorphisms generate a ring is still an open question. It is shown here that the inner-automorphisms of a group generate a ring if and only if the conjugate elements in the group commute.

Groups in which conjugate elements commute have been studied by Levi and Van der Waerden [3] and [4] in connection with the Burnside problem of exponent 3. We shall call such a group an L-group. In this note it is shown that G is an L-group, which is nilpotent of class 3 at most, if and only if the inner-automorphisms generate a ring. The endomorphisms of an abelian group form a ring. In general if the group is nonabelian the endomorphisms do not generate a ring. The characterization of groups in which the endomorphisms generate a ring is still an open question.

Theorem. The inner-automorphisms of a group G generate a ring if and only if the group is an L-group.

Proof. We consider G as a group under addition. Let A be the group of inner-automorphisms of G and R be the near-ring generated by A. Suppose R is a ring. Let a, b, g \( \in \) G, g fixed, and let u, v be the inner-automorphism of G induced by a and b respectively. We have

\[
(a + g - a) + (b + g - b) = ug + vg = (u + v)g = (v + u)g = vg + ug = (b + g - b) + (a + g - a).
\]

Hence the conjugate elements in G commute. Conversely, if G is an L-group, then the inner-automorphisms of G commute. But the inner-automorphisms of G generate the additive group \( R^+ \) of R, hence \( R^+ \) must be abelian. So R is a distributively generated near-ring with identity whose additive group is abelian and hence [2, Theorem 4.4.3] \( R \) is a ring.

Corollary. \( R \) is a commutative ring if and only if G is nilpotent of class 2.

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Proof. A group $G$ is nilpotent of class 2 if and only if the factor group with respect to the center (which is isomorphic to the group of inner-automorphisms) is abelian. But since a nilpotent group of class 2 is an $L$-group, the result follows.

References


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