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RINGS GENERATED BY THE INNER-AUTOMORPHISMS OF NONABELIAN GROUPS

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ABSTRACT. The endomorphisms of an abelian group form a ring. The characterization of groups in which endomorphisms generate a ring is still an open question. It is shown here that the inner-automorphisms of a group generate a ring if and only if the conjugate elements in the group commute.

Groups in which conjugate elements commute have been studied by Levi and Van der Waerden [3] and [4] in connection with the Burnside problem of exponent 3. We shall call such a group an L-group. In this note it is shown that G is an L-group, which is nilpotent of class 3 at most, if and only if the inner-automorphisms generate a ring. The endomorphisms of an abelian group form a ring. In general if the group is nonabelian the endomorphisms do not generate a ring. The characterization of groups in which the endomorphisms generate a ring is still an open question.

THEOREM. *The inner-automorphisms of a group G generate a ring if and only if the group is an L-group.*

PROOF. We consider G as a group under addition. Let A be the group of inner-automorphisms of G and R be the near-ring generated by A. Suppose R is a ring. Let $a, b, g \in G$, g fixed, and let u, v be the inner-automorphism of G induced by a and b respectively. We have

$$\begin{aligned}(a+g-a)+(b+g-b) &= ug+vg=(u+v)g=(v+u)g\\ &= vg+ug=(b+g-b)+(a+g-a).\end{aligned}$$

Hence the conjugate elements in G commute. Conversely, if G is an L-group, then the inner-automorphisms of G commute. But the inner-automorphisms of G generate the additive group R^+ of R, hence R^+ must be abelian. So R is a distributively generated near-ring with identity whose additive group is abelian and hence [2, Theorem 4.4.3] R is a ring.

COROLLARY. *R is a commutative ring if and only if G is nilpotent of class 2.*

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PROOF. A group G is nilpotent of class 2 if and only if the factor group with respect to the center (which is isomorphic to the group of inner-automorphisms) is abelian. But since a nilpotent group of class 2 is an L-group, the result follows.

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