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ON THE ARENS PRODUCT AND ANNIHILATOR ALGEBRAS

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ABSTRACT. The purpose of this paper is to generalize two results in a recent paper by B. J. Tomiuk and the author. Let A be a B^*-algebra and $M(A)$ the algebra of double centralizers of A. We show that A is a dual algebra if and only if $M(A)$ coincides with A^{**}. We also obtain that if A is an annihilator A^*-algebra, then $\pi_A(A)$ is a two-sided ideal of A^{**}.

1. **Notation and preliminaries.** Notation and definitions not explicitly given are taken from Rickart's book [9].

For any subset E of a Banach algebra A, let $l(E)$ and $r(E)$ denote the left and right annihilators of E in A, respectively. Then A is called an annihilator algebra if, for every closed left ideal M and for every closed right ideal N, we have $r(M)=(0)$ if and only if $M=A$ and $l(N)=(0)$ if and only if $N=A$. If $M=l(r(M))$ and $N=r(l(N))$, then A is called a dual algebra.

Let A be a Banach algebra, A^* and A^{**} the conjugate and second conjugate spaces of A, respectively. The Arens product on A^{**} is defined in stages according to the following rules (see [1]). Let x, $y \in A, f\in A^*$, F, $G\in A^{**}$.

(a) Define $f \circ x$ by $(f\circ x)(y)=f(xy)$. Then $f\circ x\in A^*$.

(b) Define $G\circ f$ by $(G\circ f)(x)=G(f\circ x)$. Then $G\circ f\in A^*$.

(c) Define $F\circ G$ by $(F\circ G)(f)=F(G\circ f)$. Then $F\circ G\in A^{**}$.

A^{**} with the Arens product $\circ$ is denoted by $(A^{**}, \circ)$.

Let A be a semisimple Banach algebra. A pair (T_1, T_2) of operators from A to A is called a double centralizer on A provided that $x(T_1y) =(T_2x)y$ for all x, y in A. It has been shown that T_1 and T_2 are continuous linear operators on A such that $T_1(xy)=(T_1x)y$ and $T_2(xy) =x(T_2y)$ for all x, y in A. The set $M(A)$ of all double centralizers on A is a Banach algebra with identity and A can be identified as a subalgebra of $M(A)$ (see [3]).

In this paper, all algebras and spaces under consideration are over the complex field C.

2. **Double centralizers of B^*-algebras.** Let A be a B^*-algebra and $(A^{**}, \circ)$ its second conjugate space with the Arens product. It is

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well known that A is Arens regular and $(A^{**}, \circ)$ is a B^*-algebra (see [**4**, p. 869, Theorem 7.1]). Since A has an approximate identity, A^{**} has an identity I by [**4**, p. 855, Lemma 3.8]. Let $M(A)$ be the algebra of double centralizers of A. By [**3**, p. 81, Theorem 2.11], $M(A)$ is a B^*-algebra with identity.

LEMMA 2.1. *Let A be a B^*-algebra. Then $M(A)$ is isometrically $*$-isomorphic to a subalgebra of $(A^{**}, \circ)$.*

PROOF. Let π be the canonical mapping of A into A^{**} and let

$$Q = \{F \in A^{**}: \pi(x) \circ F \text{ and } F \circ \pi(x) \in \pi(A) \text{ for all } x \in A\}.$$

It is clear that Q is a closed $*$-subalgebra of A^{**} containing the identity element I of A^{**}. Let $T=(T_1, T_2) \in M(A)$ and let $\{x_\alpha\}$ be an approximate identity of A. Since $\{\pi(T_1 x_\alpha)\}$ is bounded, by Alaoglu's Theorem it has weak limit points in A^{**}. Let $T_1' \in A^{**}$ be a weak limit point of $\{\pi(T_1 x_\alpha)\}$. Since $(T_1 x_\alpha)x = T_1(x_\alpha x)$, we have

$$(T_1' \circ \pi(x))(f) = f(T_1 x) = \pi(T_1 x)(f), \tag{2.1}$$

for all $f \in A^*$ and $x \in A$. Since by Goldstine's Theorem $\pi(A)$ is weakly dense in A^{**}, it follows from (2.1) and the semisimplicity of A^{**} that T_1' is unique. Similarly $\{\pi(T_2 x_\alpha)\}$ has a unique weak limit point T_2' such that $\pi(x) \circ T_2' = \pi(T_2 x)$. Since $x(T_1 y) = (T_2 x)y$ for all x, $y \in A$, it follows that $\pi(x) \circ T_1' = \pi(T_2 x)$ and $T_2' \circ \pi(x) = \pi(T_1 x)$. Now let

$$T' = \tfrac{1}{2}(T_1' + T_2') \qquad (T = (T_1, T_2) \in M(A)).$$

Then easy calculations show that $\pi(x) \circ T' = \pi(T_2 x)$ and $T' \circ \pi(x) = \pi(T_1 x)$ for all $x \in A$. Therefore $T' \in Q$. Let $S = (S_1, S_2) \in M(A)$. Since $ST = (S_1 T_1, T_2 S_2)$, we have

$$\pi(x) \circ (ST)' = \pi(T_2 S_2 x) = \pi(x) \circ (S' \circ T'),$$

and so $(ST)' = S' \circ T'$. If $T' = 0$, then $\pi(T_2 x) = \pi(x) \circ T' = 0$ and hence $T_2 = 0$. Similarly $T_1 = 0$ and therefore $T = (T_1, T_2) = 0$. It is easy to see that the mapping $T \to T'$ is an onto mapping. Therefore it is an isomorphism of $M(A)$ onto Q. Since $M(A)$ and Q are B^*-algebras, by [**7**, p. 395, Theorem B] $T \to T'$ is a $*$-isomorphism and therefore it is isometric by [**6**, p. 7, Proposition (1.3.7)] and [**6**, p. 16, Proposition (1.8.1)]. This completes the proof.

By Lemma 2.1, $M(A)$ can be considered as a subalgebra of A^{**}.

THEOREM 2.2. *Let A be a B^*-algebra. Then $M(A) = A^{**}$ if and only if A is a dual algebra.*

PROOF. This follows immediately from Lemma 2.1 and [**10**, p. 533, Theorem 5.1].

Let A be a dual commutative B^*-algebra. Then $A^{**}=A''$ by Theorem 3.2 in [**11**]. Hence Theorem 2.2 generalizes [**10**, p. 532, Theorem 4.2(3)].

3. **The Arens product and annihilator algebras.** In this section, unless otherwise stated, A will be a semisimple Banach algebra with norm $\|\cdot\|$ which is a dense subalgebra of a semisimple Banach algebra $\mathfrak{A}$ with norm $|\cdot|$. Further A and $\mathfrak{A}$ have the following properties:

(3.1) There exists a constant k such that $k\|x\| \geqq |x|$ for all $x \in A$, i.e., $\|\cdot\|$ majorizes $|\cdot|$.

(3.2) Every proper closed left (right) ideal in $\mathfrak{A}$ is the intersection of maximal modular left (right) ideals in $\mathfrak{A}$.

(3.3) A is an annihilator algebra.

If A is an A^*-algebra and $\mathfrak{A}$ is the completion of A in an auxiliary norm, then (3.1) and (3.2) automatically hold (see [**6**, p. 48, Théorème 2.9.5(iii)] and [**9**, p. 187, Corollary (4.1.16)]).

NOTATION. For any subset E of A, $\mathrm{cl}_A(E)$ (resp. $\mathrm{cl}(E)$) will denote the closure of E in A (resp. $\mathfrak{A}$).

Let S be the socle of A. Then S is dense in A by [**9**, p. 100, Corollary (2.8.16)]. Since S is contained in the socle of $\mathfrak{A}$, it follows from [**2**, p. 571, Theorem 5.2] and (3.2) that $\mathfrak{A}$ is a dual algebra. Let e be a minimal idempotent of A. Then $B=\mathrm{cl}_A(AeA)$ and $\mathfrak{B}=\mathrm{cl}(\mathfrak{A}e\mathfrak{A})$ are topologically simple, semisimple annihilator and dual algebras respectively (see [**9**, p. 100]). Let $I=Ae$ and $\mathfrak{J}=\mathfrak{A}e$. By [**9**, p. 67, Theorem (2.4.12)], B and $\mathfrak{B}$ can be considered as operator algebras on I and $\mathfrak{J}$, respectively.

LEMMA 3.1. *If E is a proper closed subspace in I, then $\mathfrak{E}=\mathrm{cl}(E)$ is a proper closed subspace in $\mathfrak{J}$.*

PROOF. Let $R=\{T\in B : T(I)\subset E\}$. Since E is a proper subspace in I, it follows easily from [**9**, p. 101, Lemma (2.8.20)] that R is a proper closed right ideal in B. Let $\mathfrak{R}=\mathrm{cl}(R)$. Then $\mathfrak{R}$ is a proper closed right ideal in $\mathfrak{B}$. Since $\mathfrak{B}$ is a dual algebra, by [**9**, p. 105, Corollary (2.8.25)], we have $\mathfrak{R}=\{\mathfrak{T}\in\mathfrak{B} : \mathfrak{T}(\mathfrak{J})\subset\mathfrak{E}'\}$, where, $\mathfrak{E}'$ is a proper closed subspace in $\mathfrak{J}$. Since B and $\mathfrak{B}$ contain all operators with finite rank on I and $\mathfrak{J}$, it follows easily that $E\subset\mathfrak{E}'$ and so $\mathfrak{E}\subset\mathfrak{E}'$. Therefore $\mathfrak{E}$ is a proper closed subspace in $\mathfrak{J}$.

Let π_A (resp. π) be the canonical mapping of A into A^{**} (resp. $\mathfrak{A}$ into $\mathfrak{A}^{**}$). A^{**} and $\mathfrak{A}^{**}$ with the Arens product will be denoted by $(A^{**}, \circ)$ and $(\mathfrak{A}^{**}, *)$.

LEMMA 3.2. *If $\pi(\mathfrak{A})$ is a two-sided ideal of $(\mathfrak{A}^{**}, *)$ then $\pi_A(A)$ is a two-sided ideal of $(A^{**}, \circ)$.*

PROOF. Let e be a minimal idempotent of A and let X_1 and X_2 be the normed spaces $(Ae, \|\cdot\|)$ and $(Ae, |\cdot|)$, respectively. The identity mapping from X_1 onto X_2 is denoted by U. It follows easily from Lemma 3.1 and (3.1) that U maps maximal closed subspaces of X_1 to maximal closed subspaces of X_2. Similarly the inverse of U has the same property. Therefore by [**8**, p. 246, Lemma B], $\|\cdot\|$ and $|\cdot|$ are equivalent on Ae. It follows easily that $Ae = \mathfrak{A}e$. Therefore we can define the linear functional $e\cdot f$ on $\mathfrak{A}$ by

$$(e\cdot f)(x) = f(xe) \qquad (x \in \mathfrak{A}, f \in A^*).$$

Since $|(e\cdot f)(x)| \leqq k_e\|f\| \, |x|$ for some constant k_e (depending on e), $e\cdot f \in \mathfrak{A}^*$. Let T be the mapping on $\pi(Ae)$ into A^{**} given by

$$(T(\pi(ye)))(f) = \pi(ye)(e\cdot f) \qquad (y \in A, f \in A^*).$$

Since $T(\pi(ye)) = \pi_A(ye)$, it follows that T is a one-one mapping of $\pi(Ae)$ onto $\pi_A(Ae)$. For each $g \in \mathfrak{A}^*$, let g_A be the restriction of g to A. Then by (3.1), $g_A \in A^*$. For each $F \in A^{**}$, let F' be the linear functional on $\mathfrak{A}^*$ defined by

$$F'(g) = F(g_A) \qquad (g \in \mathfrak{A}^*).$$

Then $F' \in \mathfrak{A}^{**}$ and hence by assumption, $F' * \pi(e) \in \pi(\mathfrak{A}e) = \pi(Ae)$. Easy calculations show that

$$(T(F' * \pi(e)))(f) = (F' * \pi(e))(e\cdot f) = (F \circ \pi_A(e))(f),$$

for all $f \in A^*$. Hence $T(F' * \pi(e)) = F \circ \pi_A(e)$ and so $F \circ \pi_A(e) \in \pi_A(Ae) \subset \pi_A(A)$. Since A has dense socle, it follows easily that $A^{**} \circ \pi_A(A) \subset \pi_A(A)$. Similarly we can show that $\pi_A(A) \circ A^{**} \subset \pi_A(A)$. This completes the proof.

The following result generalizes a part of [**10**, p. 533, Theorem 5.1].

THEOREM 3.3. *Let A be an annihilator A^*-algebra. Then $\pi_A(A)$ is a two-sided ideal of $(A^{**}, \circ)$.*

PROOF. Let $\mathfrak{A}$ be the completion of A in an auxiliary norm. Since $\mathfrak{A}$ is a dual B^*-algebra, by [**10**, p. 533, Theorem 5.1], $\pi(\mathfrak{A})$ is a two-sided ideal of $(\mathfrak{A}^{**}, *)$. Therefore by Lemma 3.2, $\pi_A(A)$ is a two-sided ideal of $(A^{**}, \circ)$.

The following corollary generalizes a result by Civin (see [**5**, p. 163, Theorem 2.4]).

COROLLARY 3.4. *Let A be the group algebra of a compact group. Then $\pi_A(A)$ is a two-sided ideal of $(A^{**}, \circ)$.*

PROOF. It is well known that A is a dual A^*-algebra. Therefore the desired result follows from Theorem 3.3.

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