PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 30, No. 1, September 1971

TAME POLYHEDRA IN WILD CELLS AND SPHERES

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ABSTRACT. It is shown that each arc on a disk D in E^4 can be homeomorphically approximated by an arc in D which is tame in E^4. Some applications of this are given. Also, we construct an everywhere wild $(n-1)$-sphere in E^n, $n \geqq 3$, each of whose arcs is tame in E^n.

1. Introduction. In [11] Seebeck showed that each arc on a disk $D^2 \subset E^n$, $n \geqq 5$, can be homeomorphically approximated by arcs in D^2 which are tame in E^n. He also showed that if $n \geqq 4$ and $2 \leqq m \leqq n-2$, then there is an m-cell in E^n which is wild at each of its points but each of whose arcs is tame in E^n.

In the current paper, we use some recent work of Bryant [3] to eliminate the missing case $n=4$ in the first result, and to sharpen the second result to the case $n \geqq 4$ and $2 \leqq m \leqq n-1$. We also give applications of the first result to problems in disk squeezing and cellularity.

We use E^n to denote Euclidean n-space and d to denote its usual metric. We sometimes regard E^n as being the $(x_{n+1}=0)$-hyperplane in E^{n+1}. In this way, if $X \subset E^n$ is compact we may regard the suspension of X, denoted ΣX, as a subset of E^{n+1}. It is the join of X with the set $\{(0, \cdots, 0, -1), (0, \cdots, 0, 1)\} \subset E^{n+1}$.

If M is a manifold we use Int M and ∂M for the interior and boundary of M respectively. We use Δ^n to denote the standard n-simplex and S^{n-1} to denote $\partial \Delta^n$.

2. **Approximating 1-complexes by tame ones.** Our main tool will be the following result due to Bryant [3, Corollary 2]. For completeness, we state it here. For convenience, we have modified its form slightly.

THEOREM 1 (BRYANT). *Let A be an arc in E^{n+1}, $n \geqq 3$. Suppose there exists a dense subset $\{t_1, t_2, \cdots\}$ of R such that $(E^n \times \{t_i\}) - A$ is 1-ULC for each i. Then A is tame.*

Received by the editors October 2, 1970.

AMS 1970 subject classifications. Primary 57A15, 57A45; Secondary 55A30, 57A60.

Key words and phrases. Everywhere wild, homeomorphically approximate, disk squeezing, cellularity.

[1] This research was supported in part by National Science Foundation Grant GP-19961.

Having set forth the tool, we may now state the theorem whose proof occupies this section.

THEOREM 2. *Let K be a finite* 1-*complex*, $h:K\to\Delta^2$ *an embedding*, $f:\Delta^2\to E^4$ *an embedding, and* $\epsilon>0$. *Then there is an embedding* $g:K\to\Delta^2$ *such that* $fg:K\to E^4$ *is tame and* $d(fg(x), fh(x))<\epsilon$ *for all* $x\in K$. *Furthermore, if L is a subcomplex of K such that $fh|L$ is tame, then g can be chosen so that* $g|L=h|L$.

PROOF. The method of proof is inductive, so we may suppose that $K-L$ consists of a single 1-simplex, say σ, with vertices a and b. We shall construct an $\epsilon/4$-push k of $(E^4, f(\Delta^2))$ and an embedding $g_\sigma:\sigma\to\Delta^2$ such that

(1) $g_\sigma|\{a,b\}=h|\{a,b\}$,

(2) $g_\sigma(\sigma)\cap h(K-\sigma)=\emptyset$,

(3) $d(kfg_\sigma(x), kfh(x))<\epsilon/2$ for all $x\in\sigma$, and

(4) kfg_σ is tame.

Then g is given by $g|K-\sigma=h|K-\sigma$ and $g|\sigma=g_\sigma$. By our assumption, fg is tame on L and, since k is a push, fg is tame on σ. Hence, fg is tame by [**4**, Added in proof]. If $x\in\sigma$, then

$$\begin{aligned} &d(fg(x), fh(x))\\ &\quad= d(k^{-1}[kfg_\sigma(x)], k^{-1}[kfh(x)])\\ &\quad\leqq d(k^{-1}[kfg_\sigma(x)], kfg_\sigma(x)) + d(kfg_\sigma(x), kfh(x)) + d(kfh(x), k^{-1}[kfh(x)])\\ &\quad< \epsilon/4+\epsilon/2+\epsilon/4=\epsilon. \end{aligned}$$

To construct k and g_σ we begin by selecting a countable dense subset $\{t_1, t_2, t_3, \cdots\}$ of R. We may suppose that $\{fh(a), fh(b)\}\subset E^4-\bigcup_{i=1}^\infty E^3\times\{t_i\}$. Let Q_i be a subpolyhedron of E^4 triangulating $E^3\times\{t_i\}$, let Q_i^r denote the rth barycentric subdivision of Q_i (with $Q_i^0=Q_i$), and let T_i^r denote the 2-skeleton of Q_i^r. Let $\epsilon_1=\epsilon/8$. By [**11**, Lemma 2], there is an ϵ_1-push k_1 of $(E^4, f(\Delta^2)\cap T_1^0)$ such that $k_1f(\Delta^2)\cap T_1^0$ is at most 0-dimensional. Let D^2 be a disk in Δ^2 such that $h(\sigma)$ spans D^2 and $\partial D^2\cap h(K)=\{h(a), h(b)\}$. There is a proper embedding $g_1:\sigma\to D^2$ such that $g_1|\{a, b\}=h|\{a, b\}$, $k_1fg_1(\sigma)\cap T_1^0=\emptyset$, and $d(k_1fg_1(x), k_1fh(x))<\epsilon_1$ for all $x\in\sigma$.

Inductively, suppose we have constructed, ϵ_i, k_i, and g_i. Let

$$\delta_i = \min\{d(k_ifg_i(\sigma), T_1^{i-1}\cup T_2^{i-2}\cup\cdots\cup T_i^0), \delta_1, \delta_2, \cdots, \delta_{i-1}\}$$

and let

$$\epsilon_{i+1} = \min\{\epsilon_i/2, \delta_i/2^{i+1}\}.$$

By [**11**, Lemma 2], there is an ϵ_{i+1}-push k'_{i+1} of $(E^4, k_if(\Delta^2)\cap(T_1^i\cup$

$\cdots \cup T_i^1 \cup T_{i+1}^0))$ such that $k'_{i+1}k_i f(\Delta^2) \cap (T_1^i \cup T_2^{i-1} \cup \cdots \cup T_i^1 \cup T_{i+1}^0)$ is at most 0-dimensional. Let $k_{i+1} = k'_{i+1}k_i$ and let $g_{i+1}: \sigma \to D^2$ be a proper embedding such that

$$g_{i+1} \mid \{a, b\} = h \mid \{a, b\},$$

$$k_{i+1} f g_{i+1}(\sigma) \cap (T_1^i \cup T_2^{i-1} \cup \cdots \cup T_i^1 \cup T_{i+1}^0) = \emptyset,$$

and

$$d(k_{i+1} f g_{i+1}(x), k_{i+1} f g_i(x)) < \epsilon_{i+1} \quad \text{for all } x \in \sigma.$$

We may also pick ϵ_{i+1} small enough (depending on k_i and g_i) so that $\{k_i\}_{i=1}^\infty$ converges to a push and $\{g_i\}_{i=1}^\infty$ converges to an embedding [7].

Let $k = \lim_{i\to\infty} k_i$ and $g_\sigma = \lim_{i\to\infty} g_i$. Then k is an $\epsilon/4$-push of $(E^4, f(\Delta^2))$. Property (1) holds since $g_i \mid \{a, b\} = h \mid \{a, b\}$ for all i, and property (2) holds since g_σ embeds σ in D^2. To see that property (3) holds, let $x \in \sigma$. Then

$$\begin{aligned} d(kfg_\sigma(x), kfh(x)) &\leqq d(k_1 f g_1(x), k_1 f h(x)) \\ &+ \sum_{i=1}^\infty d(k_i f g_i(x), k_{i+1} f g_i(x)) + \sum_{i=1}^\infty d(k_{i+1} f g_{i+1}(x), k_{i+1} f g_i(x)) < \epsilon_1 \\ &+ \sum_{i=1}^\infty \epsilon_i + \sum_{i=1}^\infty \epsilon_{i+1} = 2 \sum_{i=1}^\infty \epsilon_i \leqq \epsilon/2. \end{aligned}$$

We complete the proof by showing that kfg_σ is tame. Let $x \in \sigma$ and j, k be positive integers. Then

$$\begin{aligned} d(kfg_\sigma(x), T_j^k) &\geqq d(k_{j+k} f g_{j+k}(x), T_j^k) \\ &\quad - \sum_{i=j+k}^\infty d(k_i f g_i(x), k_{i+1} f g_{i+1}(x)) \geqq \delta_{j+k} - 2 \sum_{i=j+k+1}^\infty \epsilon_i \\ &\geqq \delta_{j+k} - 2 \sum_{i=j+k}^\infty \delta_i / 2^{i+1} \geqq \delta_{j+k} - 2 \sum_{i=j+k}^\infty \delta_{j+k}/2^{i+1} \\ &= \delta_{j+k} - 2(\delta_{j+k}/2^{j+k}) > 0. \end{aligned}$$

So $kfg_\sigma(\sigma) \cap [\bigcup_{j,k} T_j^k] = \emptyset$. Hence $(E^3 \times \{t_i\}) - kfg_\sigma(\sigma)$ is 1-ULC for all i, and, by Theorem 1, kfg_σ is tame.

REMARKS. (i) The above proof is of course valid with E^4 replaced by E^n, $n \geqq 4$. However, if $n \geqq 5$, the approach of [11] is preferred.

(ii) The theorem is valid with E^4 replaced by E^n, $n \geqq 4$, and Δ^2 replaced by a piecewise linear manifold M^m, $m \geqq 2$. This is due to the fact that any arc in M can be approximated by one which lies on a disk. In addition, we need not assume that M supports a piecewise linear structure since, by [8], some neighborhood of $h(K)$ in M does.

3. **Disk squeezing and cellularity.** The theorem stated below was proved by R. J. Daverman and W. T. Eaton for the case $n=3$ [5] and $n\geqq 5$ [6]. Theorem 2 (in place of [6, Lemma A]) allows the proof of [6, Theorem 2] to go through in the case $n=4$.

The notation of the theorem is as follows: Δ_2 is the disk in E^2 given by $\{(x, y)\,|\,x^2+y^2\leqq 1\}$, Δ_1 is the arc spanning Δ_2 and given by $\{(x, 0)\,|\,-1\leqq x\leqq 1\}$, and π is the projection of Δ_2 onto Δ_1 given by $\pi(x, y)=(x, 0)$.

THEOREM 3 (DAVERMAN AND EATON). *Suppose D^2 is a disk in E^n, $n\geqq 3$, $g_0:\Delta_2\to D^2$ is a homeomorphism, U is an open subset of E^n with $D^2-g_0(\partial\Delta_1)\subset U$ and $\epsilon>0$. Then there exists a surjective map $f:E^n\to E^n$, a homeomorphism $g:\Delta_2\to D^2$, and a homeomorphism $h:\Delta_1\to f(D^2)$ such that*

(1) $f\,|\,E^n-D^2\to E^n-f(D^2)$ *is a homeomorphism,*

(2) $f\,|\,E^n-U=\mathrm{id}_{E^n-U}$,

(3) $fg=h\pi$,

(4) $d(g(x), g_0(x))<\epsilon$ *for all* $x\in\Delta_2$, *and*

(5) $g\,|\,\partial\Delta_1=g_0\,|\,\partial\Delta_1$.

REMARK. (iii) It is interesting to note that, in case $n\neq 4$, the arcs $f^{-1}(p)$, $p\in\mathrm{Int}\,f(D^2)$, are cellular in E^n (definition below). This follows from [1, Corollary 5.5].

Applying Theorem 3 to a wedge of disks in E^n we obtain the following result.

COROLLARY 4 (DAVERMAN AND EATON). *Suppose that $P^2=(D_1^2, a)\vee(D_2^2, a)\vee\cdots\vee(D_k^2, a)\subset E^n$, $n\geqq 3$, where $a\in\partial D_i^2$ if $i=1, 2, \cdots, k$, and that U is an open subset of E^n with $P^2\subset U$. Then there exists a surjective map $f:E^n\to E^n$, homeomorphisms $g_i:\Delta_2\to D_i^2$ and homeomorphisms $h_i:\Delta_1\to f(D_i^2)$ such that*

(1) $f\,|\,E^n-P^2\to E^n-f(P^2)$ *is a homeomorphism,*

(2) $f\,|\,E^n-U=\mathrm{id}_{E^n-U}$,

(3) $fg_i=h_i\pi$ *for* $i=1, 2, \cdots, k$, *and*

(4) $g_i((-1, 0))=a=f(a)$ *for* $i=1, 2, \cdots, k$.

Recall that a continuum X in the interior of an n-manifold M^n is *cellular* if there exist n-cells $C_1, C_2, \cdots$ in M^n such that $C_{i+1}\subset\mathrm{Int}\,C_i$, $i=1, 2, \cdots$, and $X=\bigcap_{i=1}^{\infty}C_i$.

For the construction of Example 6 we require the following lemma.

LEMMA 5. *If X is a cellular continuum in E^n, then ΣX is cellular in E^{n+1}.*

PROOF. It suffices to show that if U is an open set in E^{n+1} with $\Sigma X \subset U$, then there is an $(n+1)$-cell C such that $X \subset \text{Int } C \subset C \subset U$. Since X is cellular, there exists a locally flat n-cell D in E^n such that $X \subset \text{Int } D$ and $\Sigma D \subset U$. We note that ΣX-(suspension points) lies in Int (ΣD). Now, the $(n+1)$-cell ΣD is locally flat since D is locally flat [9]. Staying in U, we obtain C from ΣD by pasting a small $(n+1)$-cell to ΣD at each suspension point.

We now construct some examples which give us an answer to the question raised in [12, Remark (viii)].

EXAMPLE 6. *There exists a noncellular k-frame in E^n, $k \geqq 3$, $n \geqq 3$, each of whose $(k-1)$-frames is cellular.*

CONSTRUCTION. For $n=3$ an explicit construction for such frames is given in [12, §4]. The frames are shown to be noncellular by showing that their complements fail to be simply connected.

Now suppose K is a k-frame in E^n with $\pi_1(E^n - K) \neq 0$ and each $(k-1)$-frame in K cellular in E^n. Let a denote the junction point of K. Since $\pi_1(E^n - K) \neq 0$, we find that $\pi_1(E^{n+1} - \Sigma K) \neq 0$. Hence ΣK fails to be cellular. But, by Lemma 5, if L is a $(k-1)$-frame in K, then ΣL is cellular in E^{n+1}.

We now squeeze ΣK to obtain the desired k-frame in E^{n+1} in two steps. First, we collapse the (tame) arc Σa to a point. This collapses ΣK to a wedge of k disks which we denote by P^2. While P^2 is not cellular, its proper sub-wedges are. Now apply Corollary 4 to squeeze P^2 to a k-frame. Again, the k-frame is not cellular, but its $(k-1)$-frames are.

REMARK. (iv) An easy proof of Theorem 2 (or its analog in E^n, $n \geqq 5$) can be given in the special case when f is a suspension embedding; this would be applicable in the above construction. For this proof, we would simply note that each arc in a suspension disk can be approximated, modulo its endpoints by one whose interior is the union of a countable collection of "horizontal" and "vertical" arcs in the product structure of $E^n = E^{n-1} \times E^1$. Such arcs are known to be tame.

4. An example in codimension one. Seebeck shows [11] that if $n \geqq 4$ and $2 \leqq m \leqq n-2$, then there is an m-cell in E^n which is wild at each of its points but each of whose arcs is tame. We show here how to use Theorem 1 to extend this result to the case $m = n-1$.

EXAMPLE 7. *There exists an $(n-1)$-sphere in E^n, $n \geqq 3$, which is wild at each of its points, but each of whose arcs is tame.*

CONSTRUCTION. If $n=3$, then Bing's "hooked rug" [2] provides our example. By suspending this example an appropriate number of times we obtain an $(n-1)$-sphere S in E^n which is wild at each point and which is the suspension of an $(n-2)$-sphere S_0 in E^{n-1}. Inductively, we may suppose that each arc in S_0 is tame in E^{n-1}. (That S is wild at each point is shown in [**10**].) Let A be an arc in S.

There is a dense subset $\{t_1, t_2, t_3, \cdots\}$ of R such that $A \cap (E^{n-1} \times \{t_i\})$ is at most 0-dimensional for each positive integer i. If $A \cap (E^{n-1} \times \{t_i\}) \neq \varnothing$, then $A \cap (E^{n-1} \times \{t_i\})$ lies on the 2-sphere $S \cap (E^{n-1} \times \{t_i\})$. (Unless, trivially, $t_i = -1$ or 1.) But the pair $E^{n-1} \times \{t_i\}$, $S \cap (E^{n-1} \times \{t_i\})$ is homeomorphic to (E^{n-1}, S_0). Hence, there is an arc $B \subset E^{n-1} \times \{t_i\}$ such that $A \cap (E^{n-1} \times \{t_i\}) \subset B$ and B is tame in $E^{n-1} \times \{t_i\}$. But then $A \cap (E^{n-1} \times \{t_i\})$ is tame in $E^{n-1} \times \{t_i\}$; hence, $E^{n-1} \times \{t_i\} - A \cap (E^{n-1} \times \{t_i\})$ is 1-ULC. By Theorem 1, A is tame.

References

1. Steve Armentrout, *Homotopy properties of decomposition spaces*, Trans. Amer. Math. Soc. **143** (1969), 499–507.

2. R. H. Bing, *A wild surface each of whose arcs is tame*, Duke Math. J. **28** (1961), 1–15. MR **23** #A630.

3. J. L. Bryant. *On embeddings with locally nice cross-sections* (to appear).

4. J. C. Cantrell, *n-frames in Euclidean k-space*, Proc. Amer. Math. Soc. **15** (1964), 574–578. MR **29** #1627.

5. R. J. Daverman and W. T. Eaton, *An equivalence for the embeddings of cells in a 3-manifold*, Trans. Amer. Math. Soc. **145** (1969), 369–381. MR **40** #3519.

6. ———, *Each disk in E^n can be squeezed to an arc* (to appear).

7. Herman Gluck, *Embeddings in the trivial range*, Ann. of Math. (2) **81** (1965), 195–210. MR **30** #3456.

8. John Hollingsworth and R. B. Sher, *Triangulating neighborhoods in topological manifolds* (to appear).

9. C. Lacher, *Locally flat strings and half-strings*, Proc. Amer. Math. Soc. **18** (1967), 299–304. MR **35** #3670.

10. T. B. Rushing, *Everywhere wild cells and spheres* (to appear).

11. Charles L. Seebeck III, *Tame arcs on wild cells* (to appear).

12. R. B. Sher, *Determining the cellularity of a 1-complex by properties of its arcs*, Proc. Amer. Math. Soc. **26** (1970), 491–498.

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