PROPERTY L AND DIRECT INTEGRAL DECOMPOSITIONS OF $W^*$ ALGEBRAS

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Abstract. If $\mathcal{A}$ is a type II$_1$ $W^*$ algebra on separable Hilbert space $H$, $\mathcal{A}$ is spatially isomorphic to $\mathcal{B} \otimes B(K)$, $\mathcal{B}$ of type II$_1$, $K$ a separable Hilbert space. If $\mathcal{A}(\lambda)$ are the factors in the direct integral decomposition of $\mathcal{A}$, the set $\mathcal{L} = \{ \lambda \mid \mathcal{A}(\lambda) \text{ has property } L \}$ is $\mu$-measurable, and $\mathcal{A}$ has property $L$ iff $\mu(\lambda - \mathcal{L}) = 0$.

Let $\mathcal{A}$ be a $W^*$ algebra on a separable Hilbert space $H$; $\mathcal{A}$ has a direct integral decomposition into factors given by

$$\mathcal{A} = \int_A^\oplus \mathcal{A}(\lambda) d\mu(\lambda).$$

Our aim is to extend [8, Theorems 4.2 and 4.3] to algebras $\mathcal{A}$ of type II$_\infty$, i.e. to algebras such that $\mathcal{A}(\lambda)$ is type II$_\infty$, $\mu$-a.e. We do this by giving a tensor product decomposition for such algebras and applying to this recent work of M. Glaser [5]. The author wishes to thank M. Glaser for making this work available.

We shall use the following notation and general results in this paper (for definitions and proofs see [8]). $K$ denotes the underlying separable Hilbert space of $H$. For a fixed sequence $\{x_i\}$ dense in the unit ball of $K$, define metrics $d_1$ and $d_2$ on $B(K)$ which induce respectively the strong and weak operator topologies on bounded subsets of $B(K)$ [7, Lemmas 1.4.8 and 1.4.9]. Let $M(\lambda) = d_1(\lambda, 0)$ and $W(\lambda) = d_2(\lambda, 0)$. Then a bounded sequence $\{\lambda_n\} \subset B(K)$ converges strongly (weakly) to 0 iff $M(\lambda_n) \to 0$ ($W(\lambda_n) \to 0$).

Let $[A, B] = AB - BA$, and let $M(A, B)$ denote $M([A, B])$. Let $S$ denote the unit ball of $B(K)$, furnished with the strong-* topology. Finally, let $B_n \subset \mathcal{A}$ be a sequence in the unit ball of $\mathcal{A}$ such that $B_n(\lambda)$ is strong-* dense in the unit ball of $\mathcal{A}(\lambda)$ $\mu$-a.e. [8, Lemma 1.5]. We may assume that $B_n(\lambda)$ is strong-* continuous in $\lambda$ (see remark following [8, Lemma 2.2]).

The following simple lemma is very useful to us.

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Lemma 1. Let $A_n \in \mathfrak{G}$ be a bounded sequence. If $A_n \to 0$ strongly (weakly) then there is a subsequence $A_{n_k}$ such that $A_{n_k}(\lambda) \to 0$ strongly (weakly) $\mu$-a.e.

Proof. By [7, Lemma 1.3.6] and the definition of $M(W)$ we know that $M(A_n(\lambda)) \to 0$ ($W(A_n(\lambda)) \to 0$) in $\mu$-measure. Hence, by [4, Corollary III.6.13] there is a subsequence $A_{n_k}$ for which this result holds $\mu$-a.e. Q.E.D.

We next introduce a generalization of the definition of central sequence given in [3]. $Z$ denotes the center of $\mathfrak{G}$ henceforth.

Definition 2. Let $A_n \in \mathfrak{G}$ be a bounded sequence. $A_n$ is central if $[A_n, A] \to 0$ strongly for all $A \in \mathfrak{G}$. $A_n$ is trivial if there is a bounded sequence $Z_n \in Z$ such that $A_n - Z_n \to 0$ strongly. $A_n$ is totally nontrivial if for each subsequence $A_{n_k}$ and each nonzero projection $E \in Z$ the sequence $EA_{n_k}E$ is not trivial. If $B_n \in \mathfrak{G}$ is another bounded sequence and $A_n - B_n \to 0$, then we say that $A_n$ and $B_n$ are equivalent.

In terms of central sequences we define Pukanszky's property $L$ [6].

Definition 3. $\mathfrak{G}$ has property $L$ if there is a unitary central sequence $U_n \in \mathfrak{G}$ such that $U_n \to 0$ weakly.

A sequence demonstrating property $L$ will be called an $L$ sequence for short. We shall soon show that an $L$ sequence contains a totally nontrivial subsequence; thus property $L$ implies the existence of a totally nontrivial central sequence (t.n.t.c.s.). First we need a preliminary lemma. We use $|\|_2$ henceforth to denote the trace norm on $\mathfrak{G}$ if $\mathfrak{G}$ is type $I_1$ [8, Definition 2.6].

Lemma 4. Let $A_n \in \mathfrak{G}$. If $A_n(\lambda)$ is central $\mu$-a.e., then $A_n$ is central. If $\mathfrak{G}$ is type $I_1$, then if $A_n$ is central there is a subsequence $A_{n_k}$ such that $A_{n_k}(\lambda)$ is central $\mu$-a.e.

Proof. If $A_n(\lambda)$ is central $\mu$-a.e., given $B \in \mathfrak{G}$ we have $M(A_n(\lambda), B(\lambda)) \to 0$ $\mu$-a.e. By [4, Corollary III.6.13] this implies $M(A_n(\lambda), B(\lambda)) \to 0$ in $\mu$-measure, whence $[A_n, B] \to 0$ strongly by [7, Lemma 1.3.6]. Hence $A_n$ is central.

If $\mathfrak{G}$ is $I_1$, note that $A_n(\lambda)$ is central in $\mathfrak{G}(\lambda)$ iff $| [A_n(\lambda), B_j(\lambda)] |_2 \to 0$ for each $j = 1, 2, \ldots$. Since $A_n$ is central, for each $j$ we have $[A_n, B_j] \to 0$ strongly, whence there is a subsequence $A_{n_k}$ such that $M(A_{n_k}(\lambda), B_j(\lambda)) \to 0$ $\mu$-a.e. by Lemma 1 and the Cantor diagonal process. Hence $| [A_{n_k}(\lambda), B_j(\lambda)] |_2 \to 0$ $\mu$-a.e. by [8, Lemma 2.12]. Q.E.D.

Lemma 5. An $L$ sequence contains a t.n.t. subsequence.
Proof. If $U_n \in \mathfrak{A}$ is an $L$ sequence, we may assume by Lemma 1 that $U_n(\lambda) \to 0$ weakly $\mu$-a.e. Note that if $Z_n \in Z$ is a bounded sequence, then by weak compactness we may assume a subsequence (again called $Z_n$) converges weakly to $Z \in Z$; since $Z_n(\lambda) = z_n(\lambda) I$ for a bounded $\mu$-measurable function, it follows that $Z_n \to Z$ strongly. Thus if $E \in Z$ is a projection such that $EU_n E - Z_n \to 0$ strongly, we may assume that for all $\lambda$ such that $E(\lambda) = I$ we have $U_n(\lambda) \to z(\lambda) I$ strongly. Thus for such $\lambda$ we have $Z(\lambda) = 0$, whence we get a contradiction since the $U_n$ are unitary. Thus $P = 0$. Q.E.D.

Now let $\mathfrak{A}$ be of type $II_{1}$ until further comment, and let $A_n \in \mathfrak{A}$ be a fixed bounded sequence. We may assume that $A_n(\lambda)$ is strong-* continuous in $\lambda$ (see remark following [8, Lemma 2.2]).

Lemma 6. The set $N = \{ \lambda | A_n(\lambda)$ is trivial $\}$ is $\mu$-measurable.

Proof. $A_n(\lambda)$ is trivial iff $| A_n(\lambda) - \text{tr}_\lambda(A_n(\lambda)) I |^2 \to 0$. If we define, for integers $j, k$, the sets

$$E(j, k) = \{ \lambda | | A_n(\lambda) - \text{tr}_\lambda(A_n(\lambda)) I |^2 \leq 1/j, n = k, k + 1, \ldots \},$$

then $N = \bigcap_{n=1}^\infty \bigcup_{k=1}^\infty \bigcap_{m=k}^\infty E(j, m)$. Each $E(j, k)$ is closed (see [8, Lemma 3.4]), whence $N$ is $\mu$-measurable. Q.E.D.

Proposition 7. $\mathfrak{A}$ has a t.n.t.c.s. iff $\mathfrak{A}(\lambda)$ has a t.n.t.c.s. $\beta$-a.e.

Proof. If $\mathfrak{A}(\lambda)$ has a t.n.t.c.s. $\mu$-a.e. then $\mathfrak{A}(\lambda)$ has property $\Gamma$ [2, Proposition 1.10] and hence property $L$ [9, Theorem 3] $\mu$-a.e. By [8, Theorem 4.3] $\mathfrak{A}$ has property $L$, and the result follows from Lemma 5. Conversely, if $A_n \in \mathfrak{A}$ is a t.n.t.c.s., we may assume by Lemma 4 that $A_n(\lambda)$ is central $\mu$-a.e. If $\mu(N) > 0$, where $N$ is as in Lemma 6, then $E \in Z$ such that $E(\lambda) = f(\lambda) I$, $f$ the characteristic function of $N$, is a projection such that $E \not= 0$ and $EA_n E$ is trivial. Hence $\mu(N) = 0$ and the result follows. Q.E.D.

Corollary 8. $\mathfrak{A}$ has a t.n.t.c.s. iff $\mathfrak{A}$ has property $L$.

We next turn our attention to algebras of type $II_\infty$. We begin by proving a structure theorem patterned on [7, Theorem II.2.22].

Theorem 9. If $\mathfrak{A}$ is type $II_\infty$, then $\mathfrak{A}$ is spatially isomorphic to $\mathfrak{A} \otimes B(J)$, where $\mathfrak{A}$ is type $II_1$ and $J$ is a separable Hilbert space.

Proof. It suffices to demonstrate the existence of a sequence of equivalent, finite, mutually orthogonal projections $E_n \in \mathfrak{A}$ such that $I = \sum_{n=1}^\infty E_n$ and to apply the argument of [7, pp. 104–105]. We produce this sequence as follows. Let $S^\infty$ denote the Cartesian product of countably many copies of $S$. $S^\infty$ is a separable metric space with
respect to the product metric arising from the strong-* topology. Let \( T = \{ x \in \mathcal{K} \mid |x| = 1 \} \). For positive integers \( j, k \), define the subset \( E(j, k) \) of \( \Lambda \times S^\infty \times S^\infty \times T \) as the set of \( (\lambda, F_n, V_n, x) \) satisfying the following conditions:

1. \( F_n, V_n \in \alpha(\lambda), n = 1, 2, \ldots \)
2. \( F_n = F_n^* = F_n^*, n = 1, 2, \ldots \)
3. \( F_1 = V_n V_n^*, F_n = V_n^* V_n, n = 1, 2, \ldots \)
4. \( F_n F_m = F_m F_n = 0, m \neq n, m, n = 1, 2, \ldots \)
5. \( F_i x = x \).
6. \( ((F_i B_m(\lambda) F_i)) (F_i B_n(\lambda) F_i) x, x) = ((F_i B_m(\lambda) F_i)) (F_i B_n(\lambda) F_i) x, x) \), \( m, n = 1, 2, \ldots \).
7. \( M(I - \sum_{i=1}^n F_i) \leq 1/j \).

Let \( R = \bigcap_{j=1}^\infty \bigcup_{k=1}^\infty E(j, k) \), and let \( \pi \) denote the projection of \( \Lambda \times S^\infty \times S^\infty \times T \) onto \( \Lambda \). It follows from \([7, \text{Lemma III.1.7}]\) and the known structure of type II\(_\infty\) factors that \( \Lambda \) differs by a \( \mu \)-null set from \( \pi(R) \). Hence by \([7, \text{Lemma I.4.7}]\) there exist \( \mu \)-measurable functions \( F_n(\lambda), V_n(\lambda) \), and \( x(\lambda) \) defined \( \mu \)-a.e. such that \( (\lambda, F_n(\lambda), V_n(\lambda), x(\lambda)) \in R \mu \)-a.e. If we put \( E_n = f_A \otimes F_n(\lambda) \mu(d\lambda) \) and recall that \( F_i(\lambda) \) (and hence each \( F_n(\lambda) \)) is finite by \([7, \text{Lemma III.1.7}]\), and that \( \sum_{n=1}^\infty E_n = I \) by \([7, \text{Lemma I.3.6}]\), we are done. Q.E.D.

We now apply Theorem 9 to extend \([8, \text{Theorems 4.2 and 4.3}]\) to \( \alpha \) of type II\(_\infty\). We remark that \([8, \text{Theorem 4.3}]\) should be stated as "if and only if", and we use this in our proof.

**Theorem 10.** Let \( \alpha \) be of type II\(_\infty\). Then \( \mathcal{L} = \{ \lambda \mid \alpha(\lambda) \text{ has property } L \} \) is \( \mu \)-measurable, and \( \alpha \) has property \( L \) if and only if \( \mu(\mathcal{L}') = 0 \), where \( \mathcal{L}' = \Lambda - \mathcal{L} \).

**Proof.** By Theorem 9, \( \alpha \) is spatially isomorphic to \( \beta \otimes B(J) \), where \( \beta \) is of type II\(_1\). By \([1, \text{Proposition II.3.3}]\) and \([7, \text{Theorem I.6.1}]\) we may assume that \( \alpha(\lambda) = \beta(\lambda) \otimes B(J) \), where \( \beta = f_A \otimes B(\lambda) \mu(d\lambda) \).

Note that a bounded sequence equivalent to t.n.t.c.s. is also t.n.t.c.s. Glaser has proved that if \( A_n \in \beta(\lambda) \otimes B(J) \) is central, then \( A_n \) is equivalent to a sequence \( B_n \otimes I \) such that \( |B_n| \leq |A_n| \) \([5, \text{Lemma 4.3}]\). Conversely, if \( B_n \otimes I \) is central, then \( B_n \otimes I \) is central \([5, \text{Theorem 5.5}]\). By these results, Lemma 5, and the initial part of the proof of Proposition 7, we see that \( \mathcal{L} = \{ \lambda \mid \beta(\lambda) \text{ has property } L \} \), whose measurability is proved in \([8, \text{Theorem 4.2}]\).

Suppose next that \( \alpha \) has property \( L \). Then \( \alpha \) and therefore \( \beta \) has a t.n.t.c.s. \([5, \text{Lemma 4.3}]\). Hence \( \mu(\mathcal{L}') = 0 \) by Corollary 8 and \([8, \text{Theorem 4.3}]\). Conversely, suppose \( \mu(\mathcal{L}') = 0 \). Then \( \alpha(\lambda) \) and hence \( \beta(\lambda) \) has a t.n.t.c.s. \( \mu \)-a.e. Hence \( \beta \) has property \( L \), and so by Lemma 4 there is an \( L \) sequence \( U_n \in \beta \) such that \( U_n(\lambda) \) is central in \( \beta(\lambda) \).
\(\mu\)-a.e. Hence, \(U_n \otimes I\) is central in \(\mathfrak{A}\) by the result of Glaser and Lemma 4. It is then clear that \(\mathfrak{A}\) has property \(L\). Q.E.D.

We remark in conclusion that, except for [8, Theorems 4.6 and 4.7], there are no results of this type known for type III algebras.

**Bibliography**


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