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APPROXIMATIONS OF THE IDENTITY OPERATOR BY SEMIGROUPS OF LINEAR OPERATORS

A. PAZY

ABSTRACT. Let $T(t)$, $t \geqq 0$, be a strongly continuous semigroup of linear operators on a Banach space X. It is proved that if for every $C>0$ there exists a $\delta_c>0$ such that $\|I-T(t)\| \leqq 2-Ct \log(1/t)$ for $0<t<\delta_c$ then $AT(t)$ is bounded for every $t>0$. It is shown by means of an example that $\|I-T(t)\| \leqq 2-Ct$ for a fixed C and all $0<t<\delta$ is not sufficient to assure the boundedness of $AT(t)$ for any $t \geqq 0$.

1. **Introduction and statement of the results.** Let X be a Banach space, $T(t)$, $0 \leqq t < \infty$, a strongly continuous semigroup of bounded linear operators on X (see Hille-Phillips [**1**, p. 321]). It is well known that the degree of approximation of the identity by the semigroup for small values of the parameter t, that is, the order of magnitude of $\|I-T(t)\|$ as a function of t, is closely related to regularity properties of the semigroup $T(t)$. Some early results concerning such approximations are given in [**1**, §10.7]. Recently J. Neuberger [**2**] proved the following result: If

$$\limsup_{t \to 0} \|I - T(t)\| < 2 \tag{1}$$

then $AT(t)$ is a bounded linear operator for every $t>0$, where A is the infinitesimal generator of $T(t)$. T. Kato [**3**] generalized this result, showing that (1) implies that $T(t)$ is a holomorphic semigroup, that is, a semigroup belonging to the class $H(\Phi_1, \Phi_2)$ for some $\Phi_1 < 0 < \Phi_2$ [**1**, p. 325] and a fortiori $AT(t)$ is bounded for every $t>0$. However, for $AT(t)$ to be bounded for every $t>0$, it is not necessary that $T(t)$ will be holomorphic. Our main result extends the result of Neuberger, showing that even if $\limsup_{t\to 0} \|I-T(t)\| = 2$ but $\|I-T(t)\|$ does not approach 2 too rapidly, $AT(t)$ is bounded for every $t>0$. Precisely our result is given by

THEOREM. *Let $T(t)$ be a strongly continuous semigroup of bounded linear operators. If for every $C>0$ there exists a $\delta_c>0$ such that*

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$$(2) \qquad \|I - T(t)\| \leqq 2 - Ct\log(1/t) \qquad \text{for } 0 < t < \delta_c,$$

then $AT(t)$ is a bounded operator for every $t>0$.

By means of Lemma 1 we can also extend the corollary to Theorem A of [2] as follows.

COROLLARY. *If $T(t)$ satisfies (2) for some fixed C and δ_c and $T(t)$ can be extended to a group of bounded linear operators, then A is bounded.*

2. **Proofs.** Our theorem as well as the corollary is a consequence of the following lemma.

LEMMA. *Let $T(t)$ be a strongly continuous semigroup of bounded operators on X satisfying $\|T(t)\| \leqq M$ for $t \geqq 0$. If there exist constants C and δ such that*

$$(3) \qquad \|I - T(t)\| \leqq 2 - Ct\log(1/t) \qquad \text{for } 0 < t < \delta,$$

then $AT(t)$ is a bounded operator for every $t > M/C$.

PROOF. Let α be real and $x \in D(A)$ (the domain of A). From the identity

$$T(t)x - e^{it\alpha}x = \int_0^t e^{i\alpha(t-s)}T(s)(A - i\alpha)x\,ds$$

it follows that $\|(T(t)-e^{it\alpha})x\| \leqq tM\|(A-i\alpha)x\|$. Substituting $\alpha = \pm\pi/t$ we obtain $\|(I+T(t))x\| \leqq tM\|(A \pm i(\pi/t))x\|$. From (3) we have

$$\|I + T(t)\| \geqq 2 - \|I - T(t)\| \geqq Ct\log(1/t) \qquad \text{for } 0 < t < \delta_c$$

and therefore

$$(4) \qquad \|(A - i\tau)x\| \geqq ((C/M)\log(|\tau|/\pi))\|x\|$$

where $\tau = \pm\pi/t$.

From (4) it follows that $A - i\tau$ is one-to-one for $|\tau|$ sufficiently large. We shall prove that it is onto X. Since A is the infinitesimal generator of a uniformly bounded semigroup $(A-(\rho+i\tau))^{-1}$ exists and has domain X for every $\rho>0$ and $\|(A-(\rho+i\tau))^{-1}\| \leqq M/\rho$ (see [4, p. 624]). Let $f \in X$ and denote by x_ρ the solution of $(A-(\rho+i\tau))x_\rho = f$. Then $\|x_\rho\| \leqq (M/\rho)\|f\|$ and therefore

$$\|(A - i\tau)x_\rho\| \leqq \rho\|x_\rho\| + \|f\| \leqq (M+1)\|f\|.$$

From (4) it then follows that $\|x_\rho\|$ is bounded as $\rho \to 0$ and

$$\|(A - i\tau)x_\rho - f\| = \rho\|x_\rho\| \to 0 \qquad \text{as } \rho \to 0$$

i.e., $(A-i\tau)x_\rho \to f$. Using (4) again we obtain that x_ρ is convergent to some element $x\in X$ as $\rho\to 0$. Since A is closed it follows that $(A-i\tau)x=f$. Thus $A-i\tau$ is onto, $(A-i\tau)^{-1}$ exists and

$$\|(A-i\tau)^{-1}\| \leqq (M/C)(\log(|\tau|/\pi))^{-1}. \tag{5}$$

Therefore,

$$\limsup_{|\tau|\to\infty}(\log|\tau|)\|(A-i\tau)^{-1}\| \leqq M/C \tag{6}$$

which implies by Theorem 2.2 of [5][1] that $AT(t)$ is bounded for $t>M/C$.

Proof of the Theorem. Let $T(t)$ be a semigroup of bounded linear operators then there exist constants M and $\omega\geqq 0$ such that $\|T(t)\|\leqq Me^{\omega t}$ [4, p. 619]. Consider the semigroup $S(t)=e^{-\omega t}T(t)$ then

$$\begin{aligned}\|S(t)-I\| &\leqq e^{-\omega t}\|T(t)-I\| + (e^{-\omega t}-1)\\ &\leqq 2 - Ct\log(1/t) + (e^{-\omega t}-1) \leqq 2 - C_1t\log(1/t)\end{aligned}$$

for every C_1, $0<C_1<C$ and $0<t<\delta(C_1)$. Thus if $T(t)$ satisfies the conditions of the Theorem (respectively of the Lemma) so does $S(t)$. But $S(t)$ is uniformly bounded and therefore differentiable for every $t>0$ (respectively $t>M/C_1$). This implies that $T(t)$ is differentiable for every $t>0$ (respectively $t>M/C_1$) which is equivalent to the boundedness of $AT(t)$ for every $t>0$ (respectively $t>M/C_1$).

Proof of the Corollary. By the previous argument $AT(t)$ is bounded for t large enough and therefore also $A=T(-t)AT(t)$ is bounded as the product of two bounded operators.

It is clear that under the conditions of our theorem we may have $\limsup_{t\to 0}\|I-T(t)\|=2$. In such a case, if $\|T(t)\|\leqq 1$ and X is uniformly convex, $T(t)$ is not holomorphic. This follows from the results of Kato [3]. Nevertheless, our theorem assures that $AT(t)$ is bounded for every $t>0$.

We conclude with an example in which $\|I-T(t)\|\leqq 2-Ct$ for some constant C and for $0\leqq t\leqq 1$ but $AT(t)$ is not bounded for any $t>0$.

Let $X=l_2$, that is, sequences $\{\alpha_n\}_{n=1}^{\infty}$ of complex numbers with the norm $\|\{\alpha_n\}\|=(\sum_{n=1}^{\infty}|\alpha_n|^2)^{1/2}$. Let

$$T(t)\{\alpha_n\} = \{\beta_n(t)\alpha_n\} \tag{7}$$

where $\beta_n(t)=\exp[(-\log(n+1)+in^n)t]$. Clearly $T(t)$ is a strongly continuous semigroup on X and

$$\|I-T(t)\| = \sup_{n\geqq 1}|1-\beta_n(t)| \leqq 1+e^{-t\log 2} \leqq 2 - C_0t, \qquad 0\leqq t\leqq 1,$$

[1] The theorem in [5] is stated with $\mu>\omega$ but it is clear from the proof that if $R(\mu+i\tau:A)$ exists for $\mu=\omega$ the result is the same. In our case $\mu=\omega=0$.

where $C_0 = \frac{1}{2}\log 2$. On the other hand,

$$\|AT(t)\| = \sup_{n \geqq 1} \left| -\log(n+1) + in^n \right| e^{-t \log n} \geqq \sup_{n \geqq 1} n^{n-t} = \infty.$$

Thus, for no $t \geqq 0$, $AT(t)$ is bounded.

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Hebrew University, Jerusalem, Israel