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CAN A 2-COHERENT PEANO CONTINUUM SEPARATE E^3?

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ABSTRACT. The fact that there are unicoherent continua which separate E^2 is well known, e.g., a circle with a spiral converging onto it is such a continuum. In this paper we extend this pathology by describing a Peano continuum which separates E^3 and has the property that however it is written as the union of two unicoherent Peano continua, their intersection is unicoherent.

An inductive definition of n-coherence has been given by Transue in [5] in such a way that 0-coherence is connectedness and 1-coherence is unicoherence plus local connectedness. A unicoherent, locally unicoherent (i.e., having a basis of unicoherent regions) set X is 2-coherent provided that however X is expressed as the union of two closed, locally connected, and unicoherent subsets A and B, the set $A\cap B$ is unicoherent.

Interesting results are obtained for 2-coherent sets: any unicoherent, locally unicoherent Peano continuum in E^3 which is not 2-coherent must separate E^3. Among the conjectures proposed by Transue are the following special cases:

CONJECTURE 1. *A 2-coherent, unicoherent continuum in E^3 does not separate E^3.*

CONJECTURE 2. *Any retract of a 2-coherent space is itself 2-coherent.*

In the introduction of [5], the author suggests that it is not known if the assumption of local unicoherence adds anything to the definition of 2-coherence. We show by an example that it does make a difference, and that the above conjectures are false without it. Let a space be called "2-coherent in the wide sense" if it satisfies the definition of 2-coherence except that the requirement of local unicoherence is omitted.

Consider the cylindrical shell $C=S^1\times[0, 1]\times[0, 1]$ in E^3, where the middle factor refers to the altitude of C and the third factor to the thickness of C. Let $C_n=S^1\times[0, 1]\times[1/2^{n+1}, 1/2^n]$, and remove from C_n an open solid rod R_n of diameter $1/2^{n+1}$, which is tangent to

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the helical line L_n where $L_n=\{(e^{2\pi nit}, t, 1/2^n):0<t\leqq 1, n$ even and $0\leqq t<1$, n odd$\}$. The boundary of R_n in C_n is a tube T_n which is capped at the bottom or top, depending on whether n is even or odd. If $Y=C\backslash\bigcup_{n=1}^{\infty} R_n$ and the two sets $S^1\times\{0\}\times\{0\}$ and $S^1\times\{1\}\times\{0\}$ are each identified to a point, a quotient space X is obtained from Y, and X is the desired space. We carry out this identification in E^3 by removing the denoted rods from C, and then pinching its top and bottom annuli so that their inner circular boundaries are shrunk to points; thus X is embedded in E^3.

Let us call the quotient map q, and note that $q(S^1\times[0, 1]\times\{0\})\equiv S$ is a 2-sphere. Moreover, S bounds one of the two complementary domains of X in E^3, and there is a retraction (the projection) of X onto S. Thus X, which we shall show to be 2-coherent in the wide sense, separates E^3 and allows a retraction onto the sphere S, which is not 2-coherent in either sense.

LEMMA 1. *If a unicoherent Peano continuum X is not 2-coherent in the wide sense, then there is an essential map $F:X\rightarrow S^2$.*

PROOF. If $X=A\cup B$, with A and B unicoherent Peano continua and $A\cap B$ is not unicoherent, then there is an essential map $f:A\cap B\rightarrow S^1$ [6, Chapter 8]. Consider S^1 as the equator of S^2 with N and Z the northern and southern hemispheres, respectively, of S^2. Then f has an extension $F_A:A\rightarrow N$ and an extension $F_B:B\rightarrow Z$ by the Tietze extension theorem. Then if $F=F_A\cup F_B$, F maps the proper triad $(X; A, B)$ into $(S^2; N, Z)$ and hence the following commutative diagram, with exact rows, exists: (Čech cohomology, integral coefficients)

$$\begin{array}{ccccc} 0 = H^1(A) + H^1(B) & \rightarrow & H^1(A\cap B) & \xrightarrow{\delta} & H^2(X) \\ & & f^*\uparrow & & F^*\uparrow \\ 0 = H^1(N) + H^1(Z) & \rightarrow & H^1(S^1) & \xrightarrow{\Delta} & H^2(S^2) \end{array}$$

Since f is essential, f^* is nonzero, and δ is 1-1 so therefore F^* is nonzero and F must be essential. Lemma 1 was proven by Transue in [5] and is included here for completeness.

THEOREM 1. *X is 2-coherent in the wide sense, and is a unicoherent Peano continuum.*

PROOF. X is clearly compact and connected. Moreover, X is locally connected since it has a basis of connected open sets in the relative topology. If X were not unicoherent, there would be a simple closed curve $J\subset X$ which is a retract of X, and hence the infinite cyclic group $H_1(J)$ would be a subgroup of $H_1(X)$. We show that $H_1(X)=0$ (Čech homology, integral coefficients) to establish that X is unicoherent. Setting $P_n = q(C\setminus\bigcup_{k=1}^n R_k)$, we see that $\{P_n : n=1, 2, \cdots\}$ is a nested system such that $\bigcap_{n=1}^\infty P_n = X$. By the continuity of Čech homology, $H_1(X)=\lim_{n\to\infty} H_1(P_n)=0$ since each P_n is a strong deformation retract of a spherical shell $\approx S^2\times[0, 1]$.

Now suppose that X were not 2-coherent in the wide sense; then $X=A\cup B$ with A and B unicoherent Peano continua and $A\cap B$ not unicoherent. We can construct the essential map $F: X\to S^2$ which was described in Lemma 1. Then F cannot be extended to a three cell containing X [**1**, p. 347] and so the homomorphism $F_*: H_2(X)\to H_2(S^2)$ is nonzero [**4**, p. 147].

We next show that the restriction of F to S maps S onto S^2. Let the set $U_n = X\setminus\bigcup_{k=1}^n q(C_k\setminus R_k)$; U_n is a neighborhood of S in X. Observe that there is a deformation retraction $r: X\to \overline{U}_n$ obtained by squashing $C_1\setminus R_1$ into $C_1\cap C_2$, and then $C_2\setminus R_2$ into $C_2\cap C_3$, and so on, a finite number of times. Thus we have a commutative diagram

$$\begin{tikzcd} H_2(X) \arrow[rr, "F_*"] \arrow[dr, "r_*"'] & & H_2(S^2) \\ & H_2(\overline{U}_n) \arrow[ur, "{[F\mid \overline{U}_n]_*}"'] & \end{tikzcd}$$

and since F_* is nonzero, $[F|\overline{U}_n]_*$ must also be nonzero. Hence $F|\overline{U}_n: \overline{U}_n\to S^2$ is an onto map for every integer n, so $F|S: S\to S^2$ is onto. Recalling the construction of F in Lemma 1, we see that neither A nor B can contain S, or else S would be mapped into one hemisphere of S^2.

Let p be a point of S which is not in A, and which is different from both the points $q(S^1\times\{1\}\times\{0\})$ and $q(S^1\times\{0\}\times\{0\})$; we call these points the "north" and "south" poles of X, respectively. Let R be a region about p which does not meet A. By the method of construction for X, for some integer N and all $n>N$, R must contain a "cross section" of every tube $q(T_n)$. Since B is unicoherent and contains R, B must contain that portion of each even numbered tube between R and the south pole of X, for $n>N$. Otherwise B could be retracted onto a simple closed curve in a tube T_n, contradicting the fact that each retract of a unicoherent Peano continuum is itself unicoherent [**6**, Chapter 8]. Similarly, B contains that portion of each odd numbered tube between R and the north pole of X, for $n>N$. However, the union of these portions of tubes mentioned is dense in $S=q(S^1\times[0, 1]\times\{0\})$, and hence B contains S, a contradiction.

Conjecture 1 remains an interesting open problem. Transue has given an affirmative answer for polyhedra in [5]. The example in this paper indicates that the technique of approximation by polyhedra may not be a useful way to attack Conjecture 1 unless local unicoherence is somehow utilized. The author has extended Transue's elegant proof that polyhedra obey Conjecture 1. Suppose that $X\subset E^3$ is a unicoherent Peano continuum, $E^3\backslash X$ has components A and B, and there is a set $T\subset X$ such that $T=h(D\times[a, b])$, h a homeomorphism, D a closed 2-cell, and a, b real numbers. If, in addition, $\mathrm{Fr}(A) \cap T = h(D \times \{a\})$, $\mathrm{Fr}(B) \cap T = h(D \times \{b\})$, and $h(D\times(a, b))$ is contained in the interior of X, then X is not 2-coherent even if $a=b$.

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