PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 30, No. 1, September 1971

TOPOLOGICAL ALGEBRAS AND MACKEY TOPOLOGIES

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ABSTRACT. Let E be a locally m-convex algebra with dual space E'. In a recent paper S. Warner asked if the finest locally m-convex topology on E compatible with E' was the mackey topology. It is shown that this is not the case. A similar result is given for this question in the A-convex algebra case. For any A-convex algebra, a construction is given of an associated locally m-convex algebra. It is shown that this associated locally m-convex topology is always the compact-open topology for the space $C_b(S)$ with the strict topology.

Seth Warner [**9**] extended the idea of bornological linear space to the case of locally m-convex algebras. For a given locally m-convex algebra E with dual space E', he noted the existence of a finest locally m-convex topology, $\chi(E, E')$, compatible with the given duality. In this note we show that $\chi(E, E')$ does not necessarily coincide with the mackey topology $\tau(E, E')$. This answers a question presented by Warner [**9**, p. 215, Question 3]. The class of A-convex algebras introduced in [**3**] and [**4**] provide a similar situation. There is a finest A-convex topology, $\Sigma(E, E')$, compatible with a given duality and we show that $\Sigma(E, E')$ is not necessarily a mackey topology.

We give a method to construct the finest locally m-convex topology coarser than a given A-convex topology. Let S denote a locally compact hausdorff space, $C_b(S)$ the space of bounded continuous complex-valued functions on S, β the strict topology introduced by Buck [**2**] and κ the compact-open topology. We use the description obtained to show that the finest locally m-convex topology coarser than β is precisely κ. Thus, there are no locally m-convex topologies between β and κ.

2. **Preliminaries.** In this section the basic definitions are given and a description of the strict topology is listed for use in §3. Throughout this note E will denote an algebra over R or C and topology will always mean locally convex linear topology.

(2.1) DEFINITION. A convex balanced absorbing subset V of E is called *m-convex* if $V\cdot V\subset V$ (i.e. if V is idempotent). A convex bal-

Received by the editors June 27, 1970.

AMS 1968 *subject classifications*. Primary 4650; Secondary 4601, 4625, 1620.

Key words and phrases. Bornological space, mackey topology, locally m-convex algebra, A-convex algebra, strict topology, compact-open topology.

anced absorbing subset V of E is called *A-convex* if, for every x in E, V absorbs xV and Vx.

(2.2) DEFINITION. A *locally m-convex algebra* is an algebra E with a topology which has a neighborhood base at zero of m-convex sets.

(2.3) DEFINITION. An *A-convex algebra* is an algebra E with a topology which has a neighborhood base at zero of A-convex sets.

For information about locally m-convex algebras see [6], [1], [8] and [9]; for A-convex algebras see [3] and [4]. An equivalent definition of A-convex algebra is the following: an A-convex algebra is an algebra E with a topology defined via a family P of seminorms such that for p in P and x in E, there are constants $M(p, x)$ and $N(p, x)$ such that

(i) $p(xy) \leqq M(p, x)p(y)$, for all y in E;

(ii) $p(yx) \leqq N(p, x)p(y)$, for all y in E.

It is clear that the class of A-convex algebras includes the class of locally m-convex algebras and, in particular, all Banach algebras.

(2.4) EXAMPLE. Let S denote a locally compact hausdorff space, $C_b(S)$ the algebra of all bounded continuous complex-valued functions on S and $C_0^+(S)$ the set of all nonnegative continuous real-valued functions on S which vanish at infinity. The strict topology, β, is defined in terms of the family of seminorms $\{p_\phi : \phi \in C_0^+(S)\}$,

$$p_\phi(f) = \sup\{ |f(x)\phi(x)| : x \in S, f \in C_b(S)\}.$$

For $S=R$, $(C_b(R), \beta)$ is a complete A-convex algebra with identity which is not locally m-convex (see [3]).

Other examples may be constructed using the generalization of Example 2.4 called weighted spaces ([3], [4] and [10]).

3. **Main results.** Let E denote an A-convex algebra with N a neighborhood base at zero consisting of A-convex sets. Warner [9] proved that the smallest idempotent set containing a given set T is $\bigcup\{T^n : n=1, 2, \cdots\}$. For each V in N, let V^* denote the balanced convex hull of $\bigcup\{V^n : n=1, 2, \cdots\}$. Since V is absorbing, V^* is m-convex. Let $N^* = \{V^* : V \in N\}$. Then N^* is a neighborhood base at zero for an m-convex topology on E.

(3.1) LEMMA. *Let (E, Φ) be an A-convex algebra with N a neighborhood base at zero of A-convex sets. Then N^* determines a locally m-convex topology Φ^* which is the finest locally m-convex topology coarser than Φ.*

PROOF. The proof is an easy consequence of the fact that V^* is the smallest m-convex set containing V.

A neighborhood base of A-convex sets for $(C_b(S), \beta)$ is given by

$$\{V(T(\phi), \theta(\phi)) : \phi \in C_0^+(S)\}$$

where $T(\phi) = \{x \in S : \phi(x) \neq 0\}$, $\theta(\phi) : T(\phi) \to R^+$ such that $\theta(\phi)(x) = 1/\phi(x)$ and $V(T(\phi), \theta(\phi)) = \{f \in C_b(S) : |f(x)| \leqq \theta(\phi)(x)$, for all $x \in T(\phi)\}$.

(3.2) THEOREM. *The associated locally m-convex topology on $C_b(S)$ for β is κ.*

PROOF. Note that $[V(T(\phi), \theta(\phi))]^* = V(T, \eta)$ where $T = \{x \in T(\phi) : \theta(\phi) \leqq 1\}$ and $\eta = \theta | T$. Then $T = \{x \in S : \phi(x) \geqq 1\}$ which is a compact subset of S since $\phi \in C_0^+(S)$. Thus, $[V(T(\phi), \theta(\phi))]^*$ is a κ-neighborhood of zero for each $\phi \in C_0^+(S)$. It is well known that $\beta \geqq \kappa$ and κ is locally m-convex. Hence the associated topology for β is κ.

(3.3) COROLLARY. *For a locally compact Hausdorff space S, there are no locally m-convex topologies on $C_b(S)$ between β and κ.*

The following result gives a solution to Warner's Question 3 of [**9**]. Let S_0 denote the space of ordinals less than the first uncountable ordinal Ω with the order topology. Conway [**5**] has shown that β is not a mackey topology on $C_b(S_0)$. Morris and Wulbert [**7**] have shown that κ is not mackey and, in fact, a result of Wang [**11**] shows that $\beta = \kappa$. Let $B = \{f \in C_b(S_0) : |f(x)| \leqq 1$ for all $x \in S_0\}$. It is known that B is not a neighborhood of zero for the mackey topology τ. In fact, for $f \in C_b(S_0)$, there exists $x_0 \in S_0$ with $f(y) = a_f$ for $y \geqq x_0$. The linear functional defined by $L(f) = a_f$ is not in the β-dual but is bounded on B. For $x \in S_0$ let $h_x(f) = f(x)$, $f \in C_b(S_0)$. Then h_x is in the β-dual. The set V defined by the closed balanced convex hull of $\{h_{x+1} - h_x : x \in S_0\}$ is weakly compact but not equicontinuous [**5**], [**7**]. Thus, V^0 is a τ-neighborhood of zero but not a κ (or β) neighborhood.

(3.4) THEOREM. *The space $(C_b(S_0), \tau)$ is not locally m-convex, where τ denotes the mackey topology compatible with κ.*

PROOF. The τ-neighborhood $W = V^0$ of zero defined above does not contain an m-convex τ-neighborhood of zero: Suppose H is m-convex, $H \subset W$. Let $f \in H$ and $x \in S_0$ with $f(x+1) \neq f(x)$. Then $|f(x)| \leqq 1$ since $f^n \in H \subset W$, $n = 1, 2, \cdots$, and $|f(x)| > 1$ gives a contradiction to $|f^n(x+1) - f^n(x)| \leqq 1$. If H is a τ-neighborhood of zero, H absorbs the τ-bounded set B. Then if $f(x) = f(x+1)$ and $|f(x)| > 1$ we use the convexity of H to obtain a function g with $|g(x)| > 1$ and $g(x)$

$\neq g(x+1)$. By the first part of the proof this is impossible. Hence $H\subset B$ so B must be a τ-neighborhood of zero. But B is not a τ-neighborhood of zero and hence W does not contain an m-convex τ-neighborhood of zero. This completes the proof.

The finite intersection of A-convex sets is an A-convex set. Thus, the supremum of A-convex topologies is A-convex. Whenever (E, Φ) is an A-convex algebra with dual E' there is a finest A-convex topology $\Sigma(E, E')$ on E which is compatible with the pair (E, E'). We now answer the obvious extension of the problem of Warner to the A-convex case.

(3.5) THEOREM. *The space* $(C_b(S_0), \tau)$ *is not* A*-convex.*

PROOF. Suppose H is an A-convex τ-neighborhood of zero with $H\subset W=V^0$. Let $x\in S_0$. Then there exists a constant K such that $|f(x)|\leqq K$ for all f in H. Suppose such a K does not exist and let $g\in C_b(S_0)$ with $|g(x+1)-g(x)|=1$ and $g(x)>1$. Then $gH\subset MH$. Using the convexity of H and the fact that H absorbs B, there exists $\sigma>0$ such that for any $L>0$ there exists $f\in H$ with $|f(x)-f(x+1)|\geqq\sigma$ and $|f(x)|>L$. This gives a contradiction to $gf\in MV^0$ for all f in H.

Let $A(x)=\inf\{M: |f(x)|\leqq M \text{ for all } f \text{ in } H\}$. Then $A(x)$ is finite for each x in S_0. Suppose A is not bounded above. Then there is a sequence $\{x_n\}$ in S_0 of distinct elements with $A(x_n)\geqq n$, $n=1, 2, \cdots$. Since S_0 is sequentially compact there is a convergent subsequence. We denote this subsequence by $\{x_k\}$ and the limit by x_0. There exists some $f\in H$ and a neighborhood of x_0, N, with $|f(x)-A(x_0)|\leqq 1$ for x in N (by continuity of f and definition of A). Also, there is an integer K such that if $n\geqq K$ then $x_n\in N$. But $\{A(x_n)\}$ is unbounded, so by convexity of H there is a function $g\in H$ such that $|g(x_n)-g(x_{n+1})|>1$ contrary to $g\in V^0$. Thus, A is bounded above so that some multiple of B contains H. This implies that B is a τ-neighborhood of zero which is a contradiction. Thus τ is not A-convex.

It is interesting to observe that for $S=R$, there are no locally m-convex topologies in the mackey spectrum of $(C_b(R), \beta)$. This result follows from Theorem 3.2 and the fact that the weak topology of $(C_b(R), \beta)$ is not locally m-convex [**4**].

We conclude this note with the following two unresolved questions.

(3.6) QUESTION. Let E be an algebra and E' a subspace of the dual (algebraic) E^*. If there are both A-convex and locally m-convex topologies compatible with (E, E') then must $\Sigma(E, E')=\chi(E, E')$?

(3.7) QUESTION. Under what conditions, in terms of E', does $\Sigma(E', E)$ and/or $\chi(E, E')$ exist?

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