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QUATERNION CONSTITUENTS OF GROUP ALGEBRAS

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ABSTRACT. In this paper it is shown that each quaternion division algebra central over the rationals appears as a division ring constituent of some rational group algebra.

Let χ be an irreducible character of a finite group and let k be a field of characteristic 0. Let $\mathcal{B}$ be the simple component of the group algebra kG corresponding to χ (i.e., such that χ takes nonzero values on $\mathcal{B}$). Then $\mathcal{B}=D_n$, a full matrix algebra over a division algebra D with center $k(\chi)$. Call *D the constituent of kG corresponding to χ*. In this paper we consider D in the case that k is the field of rational numbers Q and χ is rational-valued.

Let $m_k(\chi)$ denote the Schur index of χ over k. If D is the constituent of kG corresponding to χ, then $m_k(\chi)$ is equal to the index of D. Thus when $m_k(\chi)=2$, D has degree 4 over its center $k(\chi)$. Such a division algebra is called a *quaternion algebra*. The Brauer-Speiser Theorem (see [1, p. 165]) states that $m_k(\chi)\leqq 2$ whenever χ is real-valued. In particular when $k=Q$ and χ is rational-valued, then either $D=Q$ or D is a rational quaternion algebra. The ordinary quaternion algebra, generated over Q by the quaternions of Hamilton, occurs as a constituent corresponding to the faithful character of the quaternion group of order 8. In this paper we show that each rational quaternion algebra does occur as a constituent of some group algebra, and we give a method for constructing suitable groups which have such constituents. Henceforth D denotes a rational quaternion algebra.

THEOREM. *For each rational quaternion algebra D, there exists a finite group G such that D is a constituent of QG.*[2]

Our construction depends on the characterization of rational quaternion algebras by Hasse invariants. For rational division algebras, the Hasse invariants correspond to the set of rational primes including the infinite prime. We set the Hasse invariant

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[2] These results have also been obtained independently by K. L. Fields.

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$\mathrm{inv}_p(D)$ of D at p equal to 0 if $D \otimes_Q Q_p$ is a full matrix algebra over Q_p and equal to $\frac{1}{2}$ otherwise, where Q_p is the p-adic completion of Q at the rational prime p. Then the set of all p for which $\mathrm{inv}_p(D) \neq 0$ is a nonempty finite set of even cardinality. Conversely, for each nonempty finite set of rational primes $\{p_1, \cdots, p_{2s}\}$ of even cardinality, there exists a unique rational quaternion algebra with nonzero invariants exactly at these primes.

If D corresponds to the character χ, then $m_{Q_p}(\chi)$ equals the index of $D \otimes_Q Q_p$ for all p. When $\mathrm{inv}_p(D) = \frac{1}{2}$, $D \otimes_Q Q_p$ is a quaternion algebra with center Q_p and has index 2. Thus given a set of primes $\mathcal{P} = \{p_1, \cdots, p_{2s}\}$, we may show that the rational quaternion algebra with nonzero invariants on $\mathcal{P}$ does occur as a constituent in a group algebra by exhibiting a group G which has an irreducible rational-valued character χ with $m_{Q_p}(\chi) = 2$ exactly when $p \in \mathcal{P}$. Our task is greatly simplified by noting that if D_1, D_2, are constituents of QG_1, QG_2, respectively, then the division algebra D_3 determined by $D_1 \otimes_Q D_2$ is a constituent of $Q(G_1 \times G_2)$. Since

$$\mathrm{inv}_p(D_3) \equiv \mathrm{inv}_p(D_1) + \mathrm{inv}_p(D_2) \quad (\text{modulo } 1),$$

we need only to show the occurrence of a rational quaternion algebra with nonzero invariants exactly on the set $\{p, \infty\}$, for each prime p.

The ordinary quaternion algebra has nonzero invariants at 2 and ∞. Fix p, a finite odd prime. Suppose $p = 2^n m + 1$, where m is odd. Let A be a cyclic group of order p, T cyclic of order 2^{n+1}, and M cyclic of order m. Set $G = A(T \times M)$, the semidirect product with $A \lhd G$ and $|C_{TM}(A)| = 2$. Note that $TM/C_{TM}(A) \approx \mathrm{Aut}(A) \approx \mathrm{Gal}(Q(\epsilon_p)/Q)$, where ϵ_p is a primitive pth root of unity. Let $H = AT$ and let θ be a faithful irreducible character of H. Then θ is real-valued and $\chi = \theta^G$ is rational-valued and irreducible.

PROPOSITION. *As constructed above,* $m_{Q_q}(\chi) = 2$ *exactly when* $q = p$ *or* $q = \infty$. *In particular, the rational quaternion algebra with nonzero invariants at* p *and* ∞ *is a constituent of* QG.

PROOF. First we note that $m_k(\chi) = m_k(\theta)$ for all fields k. This follows from the fact that $m_k(\chi) \leqq 2$ and $m_k(\theta) \leqq 2$ and that $|Q(\theta):Q|$ is odd. Applying the basic inequalities available for computing Schur indices (see [2, §11]), we have

$$m_k(\chi) \mid m_k(\theta) \mid k(\theta, \chi):k(\chi)|$$

and

$$m_k(\theta) \mid m_k(\chi) \mid k(\theta, \chi):k(\theta) \mid .$$

These yield the equality of the two Schur indices.

Since $|H|$ is divisible only by 2 and p, then a result of modular theory (see [3]) yields $m_{Q_q}(\theta)=1$ if $q\notin\{2, p, \infty\}$. In order to compute $m_{Q_\infty}(\theta)$ we apply the method of Frobenius and Schur (see [2, 3.5]) to determine that θ does not have a real splitting field. Hence $m_{Q_\infty}(\theta)=2$. To compute $m_{Q_p}(\theta)$ we use a theorem of Kronstein [4] which states that since H is hyper-elementary, $m_{Q_p}(\theta)=|Q_p(\phi, \theta): Q_p(\theta)|$ where ϕ is an irreducible Brauer (p-modular) character of H which is a constituent of θ. But because A is a Sylow p-subgroup of H and $A \lhd H$, the Brauer characters of H are simply the ordinary characters of T. The irreducible Brauer characters which are constituents of H are faithful irreducible characters of T. Since $|T| = 2^{n+1}$, such a character ϕ has $Q_p(\phi)=Q_p(\epsilon_{2^{n+1}})$. Since $p=2^n m+1$, $\epsilon_{2^n}\in Q_p$ but $\epsilon_{2^{n+1}}\notin Q_p$. Also $|Q(\theta):Q|$ is odd so $\epsilon_{2^{n+1}}\notin Q_p(\theta)$. Thus $m_{Q_p}(\theta)=|Q_p(\phi,\theta):Q_p(\theta)|=2$.

Thus we have $m_{Q_q}(\chi)=2$ if $q=p$ or ∞ and $m_{Q_q}(\chi)=1$ if $q\notin\{2, p, \infty\}$. In particular $m_Q(\chi)=2$ and the constituent D of QG corresponding to χ is a rational quaternion algebra. D has nonzero Hasse invariants at p and ∞ and zero invariants elsewhere except possibly at 2. But D must have an even number of nonzero invariants so $\mathrm{inv}_2(D) = 0$. Thus $m_{Q_2}(\chi)=1$.

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