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A THEOREM ON MEAN-VALUE ITERATIONS

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ABSTRACT. In this paper we consider a function which continuously maps a closed interval of the real line into itself. It is shown that a particular mean-value iterative scheme always converges to a fixed point. The result is known for functions which have a unique fixed point. This condition is not required here.

1. **Introduction.** In this paper we consider a function f which continuously maps the closed interval $[0, 1]$ into itself. We prove that a certain mean-value iterative scheme always converges to a fixed point of f on $[0, 1]$. This result was proved in [1], where f was required to have a unique fixed point in the interval. In this paper we show that this restriction is unnecessary, convergence is proved by considering only the continuity of f: $[0, 1]\to[0, 1]$.

2. **A convergent iterative scheme.** Consider a function $f(x)$ with the following properties.

(i) $f(x)$ is continuous on $[0, 1]$.

(ii) $f(x)$ maps $[0, 1]$ into itself.

From Brouwer's fixed-point theorem, the function has at least one fixed point on this interval. We will now show that a particular mean-value iterative scheme converges to a fixed point.

THEOREM. *Let $f(x)$ continuously map the closed interval $[0, 1]$ into itself. Then the iterative scheme*

$$x_{n+1} = f(\bar{x}_n), \tag{1}$$

$$\bar{x}_n = \sum_{i=1}^{n} x_i/n, \qquad n = 1, 2, 3, \cdots, \tag{2}$$

$$\bar{x}_1 = x_1 \in [0, 1] \tag{3}$$

converges to a fixed point of $f(x)$ on $[0, 1]$.

PROOF. We first state some properties which will be useful in later developments. Combining equations (1) and (2) gives

$$\bar{x}_{n+1} = \frac{f(\bar{x}_n) - \bar{x}_n}{n + 1} + \bar{x}_n, \qquad n = 1, 2, \cdots. \tag{4}$$

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Since both $\bar{x}_n$ and $f(\bar{x}_n)$ are in $[0, 1]$, we obtain

$$(5) \qquad |\bar{x}_{n+1} - \bar{x}_n| \leqq \frac{1}{n+1}, \qquad n = 1, 2, \cdots.$$

This means the step size becomes arbitrarily small as n increases. The proof can now be accomplished in two steps.

1. We first show that $\{\bar{x}_n\}$ converges. The sequence $\{\bar{x}_n\}$ is contained in $[0, 1]$ so it has at least one limit point. For sake of contradiction assume ξ_1 and ξ_2 are two distinct limit points of $\{\bar{x}_n\}$ and $\xi_1 < \xi_2$.

a. We will show that a consequence of this assumption is that $f(x) = x$ for every x in (ξ_1, ξ_2).

Pick any $x^* \in (\xi_1, \xi_2)$. If $f(x^*) > x^*$, then by the continuity of f there is a $\delta \in (0, (x^* - \xi_1)/2)$ such that $f(x) > x$ for all x satisfying $|x - x^*| < \delta$. I.e.,

$$(6) \qquad |\bar{x}_n - x^*| < \delta \quad \text{implies} \quad \bar{x}_{n+1} > \bar{x}_n.$$

Now (5) implies there is a number N such that

$$(7) \qquad |\bar{x}_{n+1} - \bar{x}_n| < \delta, \qquad n = N, N + 1, \cdots.$$

Also $\xi_2 > x^*$ is a limit point of $\{\bar{x}_n\}$ so this N can be chosen such that $\bar{x}_N > x^*$. But, by (6) and (7),

$$\bar{x}_n > x^* - \delta > \xi_1, \qquad n = N, N + 1, \cdots,$$

which means ξ_1 is not a limit point of $\{\bar{x}_n\}$, contrary to our assumption.

If $f(x^*) < x^*$, similar reasoning contradicts the assumption that ξ_2 is a limit point. Therefore $f(x^*) = x^*$ for all $x^* \in (\xi_1, \xi_2)$.

b. We will now show that ξ_1 and ξ_2 are not both limit points. Notice that

$$(8) \qquad \bar{x}_n \notin (\xi_1, \xi_2) \quad \text{for all } n = 1, 2, \cdots$$

since if $f(\bar{x}_n) = \bar{x}_n$ then, by (4), $\bar{x}_m = \bar{x}_n$ for all $m > n$ and neither ξ_1 nor ξ_2 could be limit points.

Also (5) and (8) imply that there is a number M such that if $\bar{x}_M \geqq \xi_2$ then $\bar{x}_n \geqq \xi_2 > \xi_1$ for all $n > M$, and ξ_1 is not a limit point. If $\bar{x}_M \leqq \xi_1$ then $\bar{x}_n < \xi_1 < \xi_2$ for all $n > M$ and ξ_2 is not a limit point. Either way, $\{\bar{x}_n\}$ cannot have two distinct limit points. Therefore $\{\bar{x}_n\}$ converges to its unique limit point, ξ.

2. To show that $f(\xi) = \xi$, assume $f(\xi) > \xi$. Let $\epsilon = (f(\xi) - \xi)/2 > 0$. $\{\bar{x}_n\}$ converges to ξ and f is continuous so there is a number N such

that $f(\bar{x}_n) - \bar{x}_n > \epsilon$ for all $n > N$. By (4),

$$\bar{x}_{n+1} - \bar{x}_n = \frac{f(\bar{x}_n) - \bar{x}_n}{n+1} > \frac{\epsilon}{n+1}.$$

Therefore

$$\lim_{m\to\infty} (\bar{x}_{N+m} - \bar{x}_N) = \lim_{m\to\infty} \sum_{n=N}^{m-1} (\bar{x}_{n+1} - \bar{x}_n)$$
$$\geqq \lim_{m\to\infty} \sum_{n=N}^{m-1} \frac{\epsilon}{n+1} = \infty \cdot$$

Therefore $\bar{x}_n \to \infty$, contradicting the fact that $\bar{x}_m \in [0, 1]$ for all m.

Similarly, assuming $f(\xi) < \xi$ implies $\bar{x}_n \to -\infty$. Therefore $f(\xi) = \xi$. Q.E.D.

References

1. W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510. MR **14**, 988.

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