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# SUBHARMONIC VERSIONS OF FATOU'S THEOREM

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ABSTRACT. A theorem of Fatou concerning the boundary behavior of bounded harmonic functions in the unit disk is extended to normal subharmonic functions. A question—which has been answered in the normal holomorphic and normal harmonic cases—concerning existence of Fatou points for normal subharmonic functions is posed.

1. **Introduction.** Our starting point is the following well-known theorem of Littlewood:

THEOREM [6, p. 383]. *If $u$ is subharmonic in the unit disk $|z|<1$, and satisfies*

$$\int_0^{2\pi} |u(re^{i\theta})|\, d\theta = O(1), \qquad 0 \leqq r < 1,$$

*then* $\lim_{r\to 1} u(re^{i\theta})$ *exists* (*finitely*) *for almost all values of* $\theta$, $0\leqq\theta<2\pi$.

In 1934, Privaloff published an incorrect generalization of Littlewood's theorem in which the radial limit was replaced by angular limit. Later, as a counterexample to Privaloff's result, Zygmund described a potential function which failed to have angular limits almost everywhere on the unit circle (see [8]). Zygmund's example can be easily modified [9, p. 175] to yield a *bounded* subharmonic function with the same kind of boundary behavior. A more detailed account of this background may be found in [8].

The purpose of this paper is to prove that Privaloff's conclusion is valid if the further requirement of normality is placed on the function. Since a bounded harmonic function on $|z|<1$ is necessarily normal, we shall then be able to extend to the subharmonic case the classical theorem of Fatou concerning bounded harmonic functions in $|z|<1$.

2. **Notation and definitions.** Since its origination by Noshiro in 1938, the subject of normal meromorphic functions has been extensively developed, beginning primarily with Lehto and Virtanen [5]. However, the definition of normality is applicable to functions other

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than meromorphic ones, so, following Rung [7] we say that a real or complex-valued function $f$ defined on $|z|<1$ is a normal function provided the family

$$\mathfrak{F} = \{f(S(z)) : S \in L\},$$

where $L$ is the family of all one-to-one conformal mappings of the disk onto itself, has the property that any sequence of functions from $\mathfrak{F}$ contains a subsequence which either converges uniformly on every compact subset of $|z|<1$, or else diverges uniformly to infinity on every compact subset of $|z|<1$.

The non-Euclidean (hyperbolic) distance in $|z|<1$ is given by

$$\sigma(z_1, z_2) = \frac{1}{2} \log \frac{|1 - z_1\bar{z}_2| + |z_1 - z_2|}{|1 - z_1\bar{z}_2| - |z_1 - z_2|}.$$

For details concerning this metric see [3].

For $|\tau| = 1$ and $0 \leqq \beta < \pi/2$, let $H(\tau, \beta)$ denote the open set in $|z|<1$ bounded by the two hypercycles from $-\tau$ to $\tau$ making angle $\beta$ and $-\beta$, respectively, with the diameter through $-\tau$ and $\tau$.

Finally, the function $f$ is said to have *Fatou value* $\alpha$ at $\tau$ ($|\tau|=1$) if, for every $\beta$, $0 \leqq \beta < \pi/2$, $f(z) \to \alpha$ as $z \to \tau$ inside $H(\tau, \beta)$. In this case $\tau$ is the associated *Fatou point*. In these terms the theorem of Fatou referred to previously states that every harmonic function on $|z|<1$ which is bounded either above or below has Fatou points almost everywhere in $|z|=1$ (as a subset of $[0, 2\pi]$). In the sequel all references to measure theoretic concepts will mean Lebesgue measure on $[0, 2\pi]$.

## 3. The subharmonic version.

THEOREM. *If $u$ is normal and subharmonic on $|z|<1$ and*

$$\int_0^{2\pi} |u(re^{i\theta})| \, d\theta = O(1), \qquad 0 \leqq r < 1,$$

*then $u$ has Fatou points (corresponding to finite Fatou values) almost everywhere on the circle $|z|=1$.*

PROOF. We have the representation [6, Lemma 3, p. 390]

$$u = v + u^*,$$

where $v$ is the least harmonic majorant of $u$ in the sense that if $w_\rho(z)$ is harmonic in $|z| < \rho < 1$ and $w_\rho = u$ on $|z| = \rho$, then $\lim_{\rho \to 1} w_\rho(z) = v(z)$; $u^*$ is a nonpositive subharmonic function in $|z|<1$ with $u^*(re^{i\theta}) \to 0$

as $r \to 1$ for almost all $\theta$, $0 \leqq \theta < 2\pi$. Furthermore, Littlewood proved that

$$\int_0^{2\pi} |v(re^{i\theta})| \, d\theta = O(1), \qquad 0 \leqq r < 1,$$

so that [**9**, Theorem IV.16, p. 147] $v$ has Fatou points corresponding to finite Fatou values almost everywhere in $|z| = 1$.

Let $E$ be the set of points in $|z| = 1$ at which, simultaneously, $u^*$ has radial limit zero and $v$ has a finite Fatou value. Take $e^{i\theta}$ in $E$ (so that $v$ has Fatou value $v(\theta)$ at this point) and $\beta$ such that $0 \leqq \beta < 2\pi$. We will show that $e^{i\theta}$ is a Fatou point of $u$ with Fatou value $v(\theta)$. Let $\{z_n\}_{n=1}^{\infty} \subseteq H(e^{i\theta}, \beta)$ such that $z_n \to e^{i\theta}$ as $n \to \infty$. It suffices to prove the existence of a subsequence tending to $e^{i\theta}$ on which the function $u$ tends to $v(\theta)$.

For each $n$, $n = 1, 2, \cdots$, we denote by $E_n$ the non-Euclidean straight line which passes through $z_n$ and is also perpendicular to the radius $\rho e^{i\theta}$, $0 \leqq \rho < 1$. Label the intersection of $E_n$ with the radius to $e^{i\theta}$ by $\rho_n e^{i\theta}$. With the aid of some elementary geometry and the invariance of the metric $\sigma$ under one-to-one conformal maps of the disk onto itself, it is easily seen that each of the bounding hypercycles of $H(e^{i\theta}, \beta)$ is at hyperbolic distance $\sigma(0, \tan \beta/2)$ from the diameter through $-e^{i\theta}$ and $e^{i\theta}$. Thus

$$\sigma(\rho_n e^{i\theta}, z_n) \leqq \sigma(0, \tan \beta/2), \qquad n = 1, 2, \cdots.$$

For each $n$, $n = 1, 2, \cdots$, set

$$S_n(w) = (w + \rho_n e^{i\theta})/(1 + \rho_n e^{-i\theta} w),$$

a one-to-one conformal map of the disk onto itself.

Since $u$ is normal, there exists a subsequence, which we again denote by $\{u(S_n)\}_{n=1}^{\infty}$, which converges uniformly or else diverges uniformly on the compact set $K = \{w : \sigma(0, w) \leqq \sigma(0, \tan \beta/2)\}$. The subsequence cannot diverge uniformly on $K$ because

$$u(S_n(0)) = u(\rho_n e^{i\theta}) = v(\rho_n e^{i\theta}) + u^*(\rho_n e^{i\theta}) \to v(\theta)$$

as $n \to \infty$. Therefore, the subsequence converges uniformly on $K$ to a subharmonic function $U$.

We have

$$u(S_n(w)) \leqq v(S_n(w)), \quad \text{for } w \in K, \ n = 1, 2, \cdots,$$

and, since $e^{i\theta}$ is a Fatou point of $v$, it follows that $\{v(S_n)\}_{n=1}^{\infty}$ converges uniformly on $K$ to $v(\theta)$. This implies that $U(w) \leqq v(\theta)$, for $w \in K$.

But $U(0)=\lim_{n\to\infty} u(S_n(0))=v(\theta)$ and the Maximum Principle for subharmonic functions yields that $U\equiv v(\theta)$ in $K$. The uniform convergence of $u(S_n)$ to $v(\theta)$ on $K$ now easily yields the desired result. The set $E$ obviously has (linear) measure $2\pi$ and the proof is completed.

The following corollaries are immediate.

COROLLARY 1. *A normal subharmonic function on* $|z|<1$ *which is bounded above has Fatou points (corresponding to finite Fatou values) almost everywhere in* $|z|=1$.

PROOF. If $u$ is normal, subharmonic, and bounded above on $|z|<1$, then $v=e^u$ is normal, subharmonic, and bounded. Furthermore, every Fatou point of $v$ is a Fatou point of $u$. Thus $u$ has Fatou points almost everywhere in $|z|=1$. Arsove [1, Theorem B, p. 260] has shown that a subharmonic function bounded above on $|z|<1$ has finite radial limits almost everywhere in $|z|=1$, so our assertion concerning the finiteness of the Fatou values follows.

COROLLARY 2. *If* $u$ *is normal, subharmonic, bounded below and admits a harmonic majorant on* $|z|<1$, *then* $u$ *has Fatou points (corresponding to finite Fatou values) almost everywhere in* $|z|=1$.

PROOF. We may assume $0\leqq u(z)$, $|z|<1$. Then, for $v$ a harmonic majorant of $u$,

$$0 \leqq \int_0^{2\pi} u(re^{i\theta})d\theta \leqq \int_0^{2\pi} v(re^{i\theta})d\theta = 2\pi v(0), \qquad 0 \leqq r < 1,$$

and the conclusion follows from the theorem.

**4. Remarks.** A harmonic function on the disk which is bounded either above or below is well known to be normal. Thus normality is implicit in the statement of Fatou's theorem, so, in this sense, the corollaries in §3 represent "natural" extensions of the Fatou theorem. However, unlike the harmonic case, there is no connection, *per se*, between bounded subharmonic and normal subharmonic functions on the disk: Zygmund's counterexample (as modified by Tsuji) is a bounded nonnormal subharmonic function and, certainly, not all normal subharmonic functions are bounded.

In view of Zygmund's example we see that the normality requirement cannot be omitted from any of the results in §3. Also, Lappan [4, Corollary 2, p. 114] has constructed a normal harmonic function on $|z|<1$ for which the set of Fatou points is of linear measure zero in $|z|=1$. Thus none of the various boundedness conditions, including the harmonic majorant requirement of Corollary 2, may be dropped.

As yet unanswered questions of interest in this area are: *Must a normal subharmonic function on* $|z|<1$ *necessarily have any Fatou points? If so, is this set dense in the unit circle?*

We conclude by noting that the answer to each of these questions is in the affirmative in the holomorphic case [**2**, Corollary 1, p. 16] and also in the harmonic case [**4**, Theorem 2, p. 111].

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