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BOUNDS FOR CLASSICAL POLYNOMIALS DERIVABLE BY MATRIX METHODS

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ABSTRACT. For a matrix with dominant diagonal, convenient upper and lower bounds for the determinant are available. Such bounds occur in the work of Price, Ostrowski, Hoffman, Haynsworth, and Brenner. In this article, the inequalities are used to bound the classical polynomials over a range of values of the argument.

1. **Introduction.** Let A be a square matrix of complex numbers. Bounds on det A are known (see below) when the elements a_{ij} satisfy the hypotheses in Definitions 1.01, 1.03.

1.01. DEFINITION. $A=[a_{ij}]$ is said to have (strictly) dominant diagonal if

(1.02) $\forall_k\{|a_{kk}|>\sum_{i\neq k}|a_{ki}|=R_k\}$, i.e. the diagonal element dominates in each row.

1.03. DEFINITION. If a matrix has dominant diagonal, $\forall_k\{a_{kk}\neq 0\}$, the numbers σ_k are defined by

(1.04) $\sigma_k|a_{kk}|=R_k$.

Thus $0\leqq\sigma_k<1$.

1.05. THEOREM [4]. *Let A be an $n\times n$ matrix of complex numbers and let* (1.02) *hold. Then*

(1.06) $\prod L_k\leqq|\det A|\leqq\prod U_k$,

where $L_k=|a_{kk}|-\sum_{j>k}|a_{kj}|$, $U_k=|a_{kk}|+\sum_{j>k}|a_{kj}|$, $k=1(1)n$, $L_n=U_n=|a_{nn}|$.

1.07. THEOREM [1]. *Let A be an $n\times n$ matrix of complex numbers and let* (1.04) *hold. Then*

(1.08) $\prod L_k'\leqq|\det A|\leqq\prod U_k'$,

where $L_k'=|a_{kk}|-\sum_{j>k}\sigma_j|a_{kj}|$, $U_k'=|a_{kk}|+\sum_{j>k}\sigma_j|a_{kj}|$.

2. **Special inequalities.**

2.01. First consider the $n\times n$ matrix $A_n=[-1-2\lambda]I-\lambda(K+K^*)$, where K is the matrix with 1's on the superdiagonal and zeros elsewhere. A_n was used by Crank and Nicolson [2] to solve the heat

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equation numerically. Let $D_n=\det A_n$. The three-term recursion relation

(2.02) $D_n=(-1-2\lambda)D_{n-1}-\lambda^2 D_{n-2}$ is verified for $n=1, 2, \cdots,$ with the conventions $D_0=1, D_{-1}=0$.

Using (2.02), D_n is easily evaluated as

(2.03) $D_n=(\alpha_1^{n+1}-\alpha_2^{n+1})/(\alpha_1-\alpha_2)$,

where α_1, α_2 are the roots of the indicial equation for (2.02): $\alpha_{1,2}=\frac{1}{2}\{-1-2\lambda\pm(1+4\lambda)^{1/2}\}$.

On the other hand, the matrix A_n has dominant diagonal (satisfies (1.02)) if $\operatorname{Re}\lambda>-\frac{1}{4}$. Thus for all values of λ such that $\operatorname{Re}\lambda>-\frac{1}{4}$, application of Theorem 1.5 leads to the following corollary inequalities.

$$
\begin{aligned}
&|1+2\lambda|\{|1+2\lambda|-|\lambda|\}^{n-1}\\
&\qquad\leqq |2^{-n-1}[-1-2\lambda+(1+4\lambda)^{1/2}]^{n+1}/(1+4\lambda)^{1/2}\\
(2.04)\qquad&\qquad\qquad -2^{-n-1}[-1-2\lambda-(1+4\lambda)^{1/2}]^{n+1}/(1+4\lambda)^{1/2}|\\
&\qquad\leqq |1+2\lambda|\{|1+2\lambda|+|\lambda|\}^{n-1}.
\end{aligned}
$$

In particular if λ is real and positive, (2.04) simplifies to

$$
\begin{aligned}
&2^{n+1}(1+2\lambda)(1+\lambda)^{n-1}(1+4\lambda)^{1/2}\\
(2.05)\qquad&\qquad<(1+2\lambda+(1+4\lambda)^{1/2})^{n+1}-(1+2\lambda-(1+4\lambda)^{1/2})^{n+1}\\
&\qquad<2^{n+1}(1+2\lambda)(1+3\lambda)^{n-1}(1+4\lambda)^{1/2}.
\end{aligned}
$$

For λ in the interval $(-\frac{1}{4}, 0)$, the inequality signs in (2.05) are reversed.

2.06. *A matrix similar to A_n.* Although the $n\times n$ matrix $B_n=[\alpha_1+\alpha_2]I+\alpha_2 K+\alpha_1 K^*$ does not have dominant diagonal, nevertheless if $\arg\alpha_1=\arg\alpha_2$, the conclusions of Theorems 1.05, 1.07 do apply to this matrix, as a continuity argument shows. Actually, B_n is similar to A_n: $SB_nS^{-1}=A_n$, where $S=\operatorname{diag}[d_{11}, d_{22}, \cdots, d_{nn}]$, with $d_{jj}=(\alpha_2/\alpha_1^j)^{1/2}$.

The bounds obtainable from this matrix are less interesting; they amount to the following:

2.07. THEOREM. *If $\alpha_1>\alpha_2>0$, then (see (2.03))*

$$\alpha_1^{n-1}(\alpha_1^2-\alpha_2^2)<\alpha_1^{n+1}-\alpha_2^{n+1}<(\alpha_1+2\alpha_2)^{n-1}(\alpha_1^2-\alpha_2^2).$$

2.08. *Another matrix similar to A_n.* If more conditions are imposed on α_1, α_2, a more interesting inequality will arise. Consider the $n\times n$ matrix $C_n=(\alpha_1+\alpha_2)I+\alpha_1\alpha_2 K+K^*$. This matrix has $\alpha_1+\alpha_2$ on the diagonal, $\alpha_1\alpha_2$ on the superdiagonal, and 1 on the subdiagonal. The relation $TC_nT^{-1}=B_n$, where $T=\operatorname{diag}[\alpha_1, \alpha_1^2, \cdots, \alpha_1^n]$ shows that C_n is similar to B_n (and hence to A_n). Application of Theorem 1.05 yields

2.09. THEOREM. *Let α_1, α_2 be complex numbers such that $|\alpha_1+\alpha_2| > 1+|\alpha_1\alpha_2|$. Then*

$$\begin{aligned}&\{|\alpha_1+\alpha_2| - |\alpha_1\alpha_2|\}^{n-1}|\alpha_1^2-\alpha_2^2| \\ &\qquad \leqq |\alpha_1^{n+1}-\alpha_2^{n+1}| \leqq \{|\alpha_1+\alpha_2| + |\alpha_1\alpha_2|\}^{n-1}|\alpha_1^2-\alpha_2^2|.\end{aligned} \tag{2.10}$$

This result might be tedious to verify otherwise, especially when α_1, α_2 are not real.

2.11. *A generalization.* The matrix $E_n=T_1B_nT_1^{-1}$, where $T_1=\operatorname{diag}[1, k_2, k_2k_3, \cdots, k_2k_3\cdots k_n]$, leads to the following result.

2.12. THEOREM. *Let $\alpha_1, \alpha_2, k_2, k_3, \cdots, k_n$ be complex numbers such that the n inequalities $|\alpha_1+\alpha_2| > |\alpha_1/k_j| + |k_{j+1}\alpha_2|$ hold for $j=1(1)n$ ($k_1=\infty$, $k_{n+1}=0$). Then*

$$\begin{aligned}&|\alpha_1^2-\alpha_2^2|\prod_{j=2}^{n}\{|\alpha_1+\alpha_2| - |k_j\alpha_2|\} \\ &\qquad \leqq |\alpha_1^{n+1}-\alpha_2^{n+1}| \leqq |\alpha_1^2-\alpha_2^2|\prod_{j=2}^{n}\{|\alpha_1+\alpha_2| + |k_j\alpha_2|\}.\end{aligned}$$

Again these relations appear unpleasant to establish directly, especially so in case some of the variables are not real.

2.13. *Arbitrary diagonal elements.* To obtain the bounds of this section directly would not be difficult. But with absolutely no effort, Theorem 2.14 falls out as a corollary of Theorems 1.05 and 1.07.

2.14. THEOREM. *Let $a(n)$ be a complex-valued function of n, and let λ be a complex parameter such that*

$$|a(1)| > |\lambda|, \quad |a(j)| > |2\lambda| \quad (j = 2(1)n-1), \; |a(n)| > |\lambda|.$$

Let the numbers E_n be defined by the recursion

$$E_{-1} = 0, \quad E_0 = 1, \quad E_j = a(j)E_{j-1} - \lambda^2E_{j-2}, \qquad j = 1(1)n.$$

Then the inequalities

$$|a(1)|\prod_{j=2}^{n}(|a(j)| - |\lambda|) \leqq |E_n| \leqq |a(1)|\prod_{j=2}^{n}(|a(j)| + |\lambda|)$$

are valid bounds on $|E_n|$.

Application of Theorem 1.07 would yield more precise bounds for

$|E_n|$. Indeed, $E_n=\det\{\operatorname{diag}[a(n),\cdots,a(1)]+\lambda[K+K^*]\}$. Note that $E_1=a(1)$; $E_2=a(2)a(1)-\lambda^2$.

3. **A special polynomial.** The formula

$$\sin(n+1)\theta/\sin\theta=\det\{2\cos\theta I+K+K^*\} \tag{3.01}$$

is derivable (by induction) from a simple trigonometric identity. Furthermore it is possible to establish the relation

$$\begin{aligned}&\sin(n+1)\theta/\sin\theta\\ &=(2\cos\theta)^n-\binom{n-1}{1}(2\cos\theta)^{n-2}+\binom{n-2}{2}(2\cos\theta)^{n-4}-\cdots\\ &\quad+(-1)^p\binom{n-p}{p}(2\cos\theta)^{n-2p}+\cdots;\end{aligned} \tag{3.02}$$

see [**3**, Problems 52,53]. The algebraic identity

$$\det\{xI+K+K^*\}=\sum_{p=0}^{[n/2]}(-1)^p\binom{n-p}{p}x^{n-2p} \tag{3.03}$$

follows. If $|x|\geqq 2$, the matrix $xI+K+K^*$ has dominant diagonal. From this fact, Theorem 3.04 is a corollary.

3.04. THEOREM. *Let x be a complex number, $|x|\geqq 2$. Then the inequalities*

$$\begin{aligned}|x|\cdot(|x|-1)^{n-1}&\leqq\left|\sum_{p=0}^{[n/2]}(-1)^p\binom{n-p}{p}x^{n-2p}\right|\\ &\leqq|x|\cdot(|x|+1)^{n-1}\end{aligned}$$

hold if $n\geqq 2$.

Further, from Theorem 1.07, one obtains:

3.05. THEOREM. *If $x\geqq 2$, $n\geqq 2$ then the inequalities*

$$\begin{aligned}&(|x|^2-1)(|x|^2-2)^{n-2}/|x|^{n-2}\\ &\qquad\leqq\left|\sum_{p=0}^{[n/2]}(-1)^p\binom{n-p}{p}x^{n-2p}\right|\\ &\qquad\leqq(|x|^2+1)(|x|^2+2)^{n-2}/|x|^{n-2}\end{aligned} \tag{3.06}$$

hold.

The polynomial being estimated in (3.06) is the polynomial $\sin[(n+1)\arccos(x/2)]/\sin\arccos(x/2)$.

4. Čebyšev polynomials.

4.01. LEMMA. *The Čebyšev polynomial* $T_n(x)=\cos(n \text{ arc cos } x)$ *satisfies the three-term recursion relation* $T_n(x)=2xT_{n-1}(x)-T_{n-2}(x)$, $n\geqq 1$, *with* $T_0(x)\equiv 1$, $T_{-1}(x)\equiv 0$.

4.02. LEMMA. $T_n(x)$ *is given by the formula*

$$T_n(x) = \det\{K + K^* + \operatorname{diag}[2x, \cdots, 2x, x]\}.$$

PROOF. The determinant has the correct starting values and satisfies the correct three-term recursion.

From Theorems 1.05, 1.07 the estimates below are obtained.

4.03. THEOREM. *Let* $|x|\geqq 1$, $n\geqq 1$. *Then*

$$[2|x| - 1]^{n-1}|x| \leqq |T_n(x)| \leqq [2|x| + 1]^{n-1}|x|. \tag{4.04}$$

4.05. THEOREM. *Let* $|x|\geqq 1$, $n\geqq 2$. *Then*

$$\begin{aligned}(2|x|^2-1)^{n-1}/|x|^{n-2} &\leqq |T_n(x)| \\ &\leqq (2|x|^2+1)^{n-1}/|x|^{n-2}.\end{aligned} \tag{4.06}$$

Estimates for $|T_n(x)|$ are well known when x is real, $|x|\leqq 1$. The estimates given by Theorems 4.04, 4.06 are reasonable; they grow about as fast as a polynomial of degree n, which $T_n(x)$ certainly is: $T_n(x)=2^{n-1}x^n+\cdots$.

5. **Hermite polynomials.** Let $H_n=(-1)^n e^{x^2/2}D^n e^{-x^2/2}$ be the nth Hermite polynomial, so that $H_0=1$, $H_1=x$, $H_2=x^2-1$, $H_3=x^3-3x$, $H_4=x^4-6x^2+3$, $H_5=x^5-10x^3+15x$. Then

(5.01) $H_{n+1}(x)=xH_n(x)-nH_{n-1}(x)$.

Therefore

(5.02) $H_n(x)=\det[xI+K^*+W_4]$, where W_4 has superdiagonal $n-1$, $n-2$, $\cdots$, 1 and 0's elsewhere. This gives the following bounds.

5.03. THEOREM. *For* $|x|>n>0$, *the Hermite polynomial* $H_n(x)=(-1)^n e^{x^2/2}D^n e^{-x^2/2}$ *satisfies the following inequalities*:

$$\prod_{j=0}^{n-1}(|x| - j) \leqq |H_n(x)| \leqq \prod_{j=0}^{n-1}(|x| + j), \tag{5.04}$$

$$(5.05)\qquad \begin{aligned}&(|x|^2-n+1)\prod_{j=1}^{n-1}(|x|^2-j)\\ &\qquad \leqq |x|^n|H_n(x)| \leqq (|x|^2+n-1)\prod_{j=1}^{n-1}(|x|^2+j).\end{aligned}$$

Bounds for a larger range of values of x are obtained by using the matrix $M=xI+W_5+W_5^*$, where each element of W_5 is the square root of the corresponding element of W_4; this matrix is similar to the preceding one.

5.06. THEOREM. *Let* $|x|>(n-1)^{1/2}+(n-2)^{1/2}$, $n>1$. *Then* $H_n(x) = (-1)^n e^{x^2/2}D^n e^{-x^2/2}$ *is bounded as follows.*

$$(5.07)\qquad \prod_{j=0}^{n-1}(|x|-\sqrt{j}) \leqq |H_n(x)| \leqq \prod_{j=0}^{n-1}(|x|+\sqrt{j})$$

$$(5.08)\qquad \begin{aligned}&(|x|^2-1)\prod_{j=2}^{n-1}(|x|^2-2j)/|x|^{n-2}\\ &\qquad \leqq |H_n(x)| \leqq (|x|^2+1)\prod_{j=2}^{n-1}(|x|^2+2j)/|x|^{n-2}.\end{aligned}$$

5.09. REMARK. Note that these bounds are in general better than the bounds of the preceding theorem, but they are invalid for $n=1$.

5.10. The recursion formula for

$$K_n(x) = nH_n(x) = (-1)^n n e^{x^2/2}D^n e^{-x^2/2}$$

is

$$(5.11)\quad K_{n+1}(x) = x\left(1+\frac{1}{n}\right)K_n(x) - \left(n+2+\frac{2}{n-1}\right)K_{n-1}(x),$$

which is asymptotically like the recursion formula (5.01) for $H_n(x)$. This gives bounds for $K_n(x)$, and hence for $H_n(x)$, that are better than the preceding bounds, for some moderately large values of x. Since the bounds are, for very large x, inferior to the preceding, the details are suppressed.

6. **Legendre polynomials.** The polynomials

$$M_n(x) = 2^{-n}(n!)^{-1}D^n(x^2-1)^n$$

satisfy the recursion $(n>0)$

$$(6.01)\qquad \begin{aligned}nM_n(x) &= (2n-1)xM_{n-1}(x) - (n-1)M_{n-2}(x),\\ &\qquad M_0 = 1,\ M_{-1} = 0\end{aligned}$$

Therefore $M_0=1$, $M_1=x$, $M_2=\frac{1}{2}(3x^2-1)$, $M_3=\frac{1}{2}(5x^3-3x)$, $M_4=\frac{1}{8}(35x^4-30x^2+3)$; $M_n(x)=\det\{xW_6+W_7+W_7^*\}$, where

$$W_6 = \text{diag}[2-n^{-1}, 2-(n-1)^{-1}, \cdots, 2-1],$$

$$W_7 = \text{super diag}[1-n^{-1}, \cdots, 1-2^{-1}].$$

6.02. THEOREM. *Let* $n \geqq 2$. *Then* $M_n(x)=2^{-n}(n!)^{-1}D^n(x^2-1)^n$ *is bounded as follows for* $|x| \geqq 4/3$.

$$(6.03)\qquad \begin{aligned} &|x|\prod_{j=2}^{n}\{(2-j^{-1})|x| - (1-j^{-1})^{1/2}\} \\ &\leqq |M_n(x)| \leqq |x|\prod_{j=2}^{n}\{(2-j^{-1})|x| + (1-j^{-1})^{1/2}\}. \end{aligned}$$

The range of validity of these inequalities shrinks down towards $|x|>1$ for sufficiently large n.

7. **Laguerre polynomials.** The Laguerre polynomials are defined by

$$(7.01)\qquad \begin{aligned} L_n(x) &= (-1)^n e^x \frac{d^n}{dx^n} x^n e^{-x} \\ &= x^n - n^2x^{n-1} + \frac{n^2(n-1)^2}{2!}x^{n-2}\cdots \pm n! \end{aligned}$$

They satisfy the recursion relation $L_{n+1}=(x-2n-1)L_n-n^2L_{n-1}$ $(n\geqq 0)$; $L_0=1$.

Therefore $L_n(n>1)$ is given by $L_n(x)=\det\{W_8+W_9+W_9^*\}$, where

$$W_8 = \text{diag}[x-2n+1, x-2n+3, \cdots, x-3, x-1],$$

$$W_9 = \text{super diag}[n-1, n-2, \cdots, 2, 1].$$

This leads to the following estimates.

7.02. THEOREM. *For* $n>3$, *if* $|x-2n+3| \geqq 2n-3$, *then* $L_n(x) = (-1)^n e^x D^n(x^n e^{-x})$ *satisfies*

$$(7.03)\qquad \begin{aligned} &|x-1|\prod_{j=1}^{n-1}\{|x-2j-1| - j\} \\ &\leqq |L_n(x)| \leqq |x-1|\prod_{j=1}^{n-1}\{|x-2j-1| + j\}. \end{aligned}$$

7.04. THEOREM. *For* $n>3$, *if* $|x-2n+3| \geqq 2n-3$, *then* $L_n(x) = (-1)^n e^x D^n(x^n e^{-x})$ *satisfies*

$$
\begin{aligned}
&| x^2 - 4x + 2 | \prod_{j=2}^{n} | x^2 - 4jx + 2j^2 + j - 1 | \\
(7.05)\qquad &\leqq | L_n(x) | \cdot \prod_{j=2}^{n-1} | x - 2j + 1 | \\
&\leqq | x^2 - 4x + 4 | \prod_{j=2}^{n} | x^2 - 4jx + 6j^2 - j - 1 |.
\end{aligned}
$$

In particular, these inequalities are valid for all nonpositive x $(x \leqq 0)$. The lower bound for $|L_n(0)|$ is exact.

8. **Generalized Laguerre polynomials.** The modified Laguerre polynomial is

$$(8.01)\qquad \overset{a}{L}_n(x) = n!^{-1} e^{x} x^{-a} D^{n}(e^{-x} x^{n+a});$$

$$
(8.02)\qquad
\begin{aligned}
&\overset{a}{L}_{-1} = 0, \qquad \overset{a}{L}_0 = 1, \qquad \overset{a}{L}_1 = -x + 1 + a, \\
&\overset{a}{L}_2 = x^2 - 2(2 + a)x + (2 + a)(1 + a), \cdots .
\end{aligned}
$$

These polynomials satisfy the recursion

$$(8.03)\qquad (n + 1)\overset{a}{L}_{n+1} = (2n + a + 1 - x)\overset{a}{L}_n - (n + a)\overset{a}{L}_{n-1}, \qquad n \geqq 0.$$

In (8.02), (8.03), a can be any complex number. In Theorem 8.05, a is real. Thus $L_n^a(x) = \det\{ W_{10} + W_{11} + W_{11}^* \}$, where

$$
(8.04)\qquad
\begin{aligned}
W_{10} &= \operatorname{diag}[n^{-1}(2n + a - 1 - x), (n - 1)^{-1}(2n + a - 3 - x), \\
&\qquad \cdots, 2^{-1}(a + 3 - x), a + 1 - x], \\
W_{11} &= \text{super diag}[n^{-1/2}(n + a - 1)^{1/2}, (n - 1)^{-1/2}(n + a - 2)^{1/2}, \\
&\qquad \cdots, 2^{-1/2}(a + 1)^{1/2}].
\end{aligned}
$$

If $a \geqq 4/3$, $n \geqq 4$, the diagonal of this matrix dominates for all x such that $|2n+a-3-x| \geqq 2((n-1)(n+a-2))^{1/2}$.

8.05. THEOREM. *Let $a \geqq 4/3$, $n \geqq 4$. For all x such that $|2n+a-3-x| \geqq 2((n-1)(n+a-2))^{1/2}$, the modified Laguerre polynomial $L_n^a(x) = n!^{-1}e^{x}x^{-a}D^{n}(e^{-x}x^{n+a})$ satisfies*

$$
\begin{aligned}
&| a + 1 - x | \prod_{j=2}^{n} \{ | a - 1 - x + 2j | - ((j + a - 1)/j)^{1/2} \} \leqq | \overset{a}{L}_n(x) | \\
&\qquad \leqq | a + 1 - x | \prod_{j=2}^{n} \{ | a - 1 - x + 2j | + ((j + a - 1)/j)^{1/2} \}.
\end{aligned}
$$

9. **Arctangent polynomials.** The polynomials $S_n(x)$ defined by

$$n!S_n(x) = (-1)^n(1 + x^2)^{n+1}D^n(1 + x^2)^{-1} \tag{9.01}$$

satisfy the recursion ($n>0$) [3]

$$S_n = 2xS_{n-1} - (x^2 + 1)S_{n-2}, \qquad S_{-1} = 0,\ S_0 = 1. \tag{9.02}$$

For a small range of values of Re x, Im x, bounds for these polynomials can be obtained. The $n\times n$ matrix ($n>0$) $W_{12}=2\times I+(x^2+1)^{1/2}K$ has dominant diagonal if $|x|\geqq|(x^2+1)^{1/2}|$, i.e. if $2[\text{Im } x]^2\geqq(1+2[\text{Re } x]^2)/(1-[\text{Re } x]^2)$.

9.03. THEOREM. *For* $n>0$, $|x^2|\geqq|x^2+1|$, *the polynomial* $S_n(x)=(-1)^n(n!)^{-1}(1+x^2)^{n+1}D^n(1+x^2)^{-1}$ *satisfies the relations*:

$$|2x|(|2x| - |(x^2+1)^{1/2}|)^{n-1} \leqq |S_n(x)| \leqq |2x|(|2x| + (x^2+1)^{1/2})^{n-1};$$

$$2\{|2x^2| - |x^2+1|\}^{n-2}\{|4x^2| - |x^2+1|\} \leqq |x^{n-2}S_n(x)| \leqq 2\{|2x^2| + |x^2+1|\}^{n-2}\{|4x^2| + |x^2+1|\}.$$

9.04. *Arcsinh polynomials.* The polynomials

$$U_n(x) = (-1)^n(1 + x^2)^{n+1/2}D^n(1 + x^2)^{-1/2} \tag{9.05}$$

satisfy the recursion ($n>0$)

$$U_n = (2n-1)xU_{n-1} - (n-1)^2(1+x)^2U_{n-2}, \qquad U_0 = 1,\ U_{-1} = 0. \tag{9.06}$$

Therefore $U_n(x)=\det\{W_{14}+W_{13}+W_{13}^*\}$, where

$$W_{14} = x\,\text{diag}[2n-1, 2n-3, \cdots, 3, 1]; \qquad W_{13} = (1+x^2)^{1/2}W_9. \tag{9.07}$$

9.08. THEOREM. *For* $n>0$, $|x^2|\geqq|x^2+1|$, *the polynomial* $U_n(x)=(-1)^n(1+x^2)^{n+1/2}D^n(1+x^2)^{1/2}$ *satisfies the inequalities*

$$\begin{aligned} |x|\{(2n-1)|x| - (n-1)|(1+x^2)^{1/2}|\}\{|x| - |(1+x^2)^{1/2}|\}^{n-2} &\leqq |U_n(x)|/\prod_{j=1}^{n-2}(2j+1) \\ &\leqq |x|\{(2n-1)|x| + (n-1)|(1+x^2)^{1/2}|\} \cdot\{|x| + |(1+x^2)^{1/2}|\}^{n-2}. \end{aligned} \tag{9.09}$$

9.10. *Polynomials derived from* $\exp(1/x)$. The polynomials V_n defined by (see [5])

$$(9.11) \qquad V_n(x) = (-1)^{n+1}x^{2n+2}\exp(-1/x)D^{n+1}\exp(1/x)$$

satisfy the recursion ($n>0$)

$$(9.12) \qquad V_n = (2nx+1)V_{n-1} - n(n-1)x^2V_{n-2}.$$

Therefore $V_n(x) = \det\{I + 2xW_{16} + xW_{15} + xW_{15}^*\}$, where $W_{16} = \operatorname{diag}[n, n-1, \cdots, 1]$, $W_{15} = \text{super diag}[n, n-1, \cdots, 2]$. For $n>1$, this matrix has dominant diagonal if Re $x \geqq -1/(4n-4)$, in particular if Re $x \geqq 0$. Bounds are easily derived; details are suppressed.

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