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A NOTE ON SOME OPERATOR THEORY IN CERTAIN SEMI-INNER-PRODUCT SPACES

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ABSTRACT. Let X be a smooth uniformly convex Banach space and let $[\cdot,\cdot]$ be the unique semi-inner-product generating the norm of X. If A is a bounded linear operator on X, $A^\dagger$ mapping X to X is called the generalized adjoint of A if and only if $[A(x), y] = [x, A^\dagger(y)]$ for all x and y in X. In this setting adjoint abelian (iso abelian) operators [5] are characterized as those operators A for which $A^\dagger = A$ ($A^\dagger = A^{-1}$, i.e. the invertible isometries). It is also shown that the compression spectrum of an operator is contained in its numerical range.

In a recent paper Giles [1] has shown that in a fairly large class of Banach spaces it is possible to construct a semi-inner-product possessing some of the desirable properties of the Hilbert space inner product. (See Lumer [4] for a discussion of semi-inner-products.) In particular he has considered the class of smooth uniformly convex Banach spaces and he has shown that in these spaces the (unique) semi-inner-product has the following properties.

(0.1) $[x, \lambda y] = \bar{\lambda}[x, y]$ for all scalars λ.

(0.2) $[x, y] = 0$ if and only if y is orthogonal to x, i.e., if and only if $\|y\| \leqq \|y + \lambda x\|$ for all scalars λ.

(0.3) (The generalized Riesz-Fischer representation theorem.) If f is a continuous linear functional on X then there is a unique vector y in X such that $f(x) = [x, y]$ for all x in X. Hence $[x, y] = [x, z]$ for all x in X if and only if $y = z$.

The purpose of this note is to explore several interesting aspects of the operator theory possible in this setting, and to show how it is related to the work of Stampfli on adjoint abelian and iso abelian operators [5].

For the rest of the paper we shall assume that X is a smooth uniformly convex Banach space with the semi-inner-product described above.

I. **The numerical range.** Recall that in a semi-inner-product space

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X the numerical range of a bounded linear operator A is $W(A) = \{[Ax, x]: \|x\| = 1\}$.

THEOREM 1. *The compression spectrum of a bounded linear operator is contained in its numerical range.*

PROOF. If t is in the compression spectrum then the range of $(tI-A)$ is not dense. Then it is easy to find a vector y such that y is orthogonal to the range of $(tI-A)$. We can pick y so that $\|y\|=1$. From (0.2) we get $0=[(tI-A)(y), y]=t-[A(y), y]$. As a result $t=[A(y), y]$ is in the numerical range of A.

Since Lumer has shown that if $(tI-A)$ is not bounded below then t is in the closure of the numerical range [**4**], it follows that the spectrum is contained in the closure of the numerical range. (See Williams [**5**] for a proof of this fact in an arbitrary Banach space.) Lumer has also shown that the numerical range need not be convex. It would be interesting to know if it were convex in this setting.

II. **The generalized adjoint.** If A is a bounded linear operator from X to itself then g_y, defined by $g_y(x)=[A(x), y]$, is a continuous linear functional, and from (0.3) it follows that there is a unique vector $A^{\dagger}(y)$ such that $[A(x), y]=[x, A^{\dagger}(y)]$ for all x in X. We shall call $A^{\dagger}$ the *generalized adjoint* of A.

In the case X is a Hilbert space the generalized adjoint is the usual Hilbert space adjoint. In our more general setting this map is not usually linear but it still has some interesting properties. We first introduce some notation that is useful. We let ϕ be a map from X to X^* given by $\phi(y)=f_y=[\cdot, y]$. Then ϕ is a duality map in the sense of Stampfli [**5**]. (0.3) shows ϕ is one-to-one and onto, and ϕ^{-1} is always continuous (X is uniformly convex). ϕ will be continuous at a point x in X if and only if the norm of X is strongly (Fréchet) differentiable at that point [**2**].

THEOREM 2. (a) $(\lambda A)^{\dagger}=\bar{\lambda}A^{\dagger}$.

(b) $(AB)^{\dagger}=B^{\dagger}A^{\dagger}$.

(c) *$A^{\dagger}$ is one-to-one if and only if the range of A is dense in X.*

(d) $A^*\phi=\phi A^{\dagger}$.

(e) *If norm of X is strongly* (*Fréchet*) *differentiable then $A^{\dagger}$ is continuous.*

PROOF. The proofs of the first three properties are the usual Hilbert space proofs using (0.1), (0.2), and (0.3).

Since $A^*\phi(x)(y)=[A(y), x]=[y, A^{\dagger}(x)]=\phi A^{\dagger}(x)(y)$, (d) is true.

The last fact follows from (d) and the fact that in this case ϕ is a homeomorphism.

Stampfli has defined a bounded linear operator A to be *adjoint abelian* if and only if there is a duality map ϕ such that $A^*\phi=\phi A$ [**5**]. Thus in this setting we have the following corollary.

COROLLARY 1. *A is adjoint abelian if and only if $A=A^\dagger$.*

This indicates that the adjoint abelian operators are in some sense "selfadjoint".

III. **Invertible isometries.** We now turn to a study of the invertible isometries in these spaces.

THEOREM 3. *In a smooth Banach space U is an isometry if and only if it preserves the semi-inner-product.*

PROOF. If X is smooth, then given any x in X there is a unique x^* in X^* such that $\|x^*\|=\|x\|$ and $x^*(x)=\|x\|^2$. Hence the semi-inner-product is unique. But then if U is any isometry, $\langle x, y\rangle=[U(x), U(y)]$ defines a semi-inner-product, and by the uniqueness we have $[U(x), U(y)]=[x, y]$.

Conversely if $[x, y]=[U(x), U(y)]$ for all x and y in X then $\|x\|^2=[x, x]=[U(x), U(x)]=\|U(x)\|^2$. Hence U is an isometry.

COROLLARY 2. *In any smooth uniformly convex Banach space, U is an invertible isometry if and only if $U^{-1}=U^\dagger$. As a result if in addition $U^{-1}=U$ then U is scalar.*

PROOF. The proof of the characterization is now the usual Hilbert space argument. If in addition $U^{-1}=U$, it follows that $U=U^\dagger$. Thus U is an invertible adjoint abelian operator. Stampfli has shown that all such operators are scalar [**5**].

Finally, Stampfli has defined an operator to be *iso abelian* if and only if there is a duality map ϕ such that $\phi U=(U^*)^{-1}\phi$.

COROLLARY 3. *In a smooth Banach space U is iso abelian if and only if it is an invertible isometry.*

(Theorem 3 and Corollary 3 have been extended to include the nonsmooth case in [**3**].)

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