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A v-INTEGRAL REPRESENTATION FOR LINEAR OPERATORS ON A SPACE OF CONTINUOUS VECTOR-VALUED FUNCTIONS

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ABSTRACT. In this note an analytic representation is given for continuous linear operators from $C(X)$ into a linear normed space Y where $C(X)$ is the space of continuous functions on $[0, 1]$ with values in a linear normed space X.

1. **Introduction.** With a very elaborate and difficult construction D. H. Tucker [3] obtains a Stieltjes-type integral representation for continuous linear operators from $C(X)$ into a linear normed space Y, where $C(X)$ is the space of continuous functions on $[0, 1]$ with values in a linear normed space X. In this note we give a representation theorem in terms of the v-integral (introduced in [1]) and the argument is straightforward.

2. **The representation theorem.** Suppose K is a set function defined on half open intervals $(a, b]\subset[0, 1]$ with values in $B[X, Y]$, the space of bounded linear operators from X into Y. If there is a constant M such that for any disjoint collection of such intervals $\{I_i\}$ and any corresponding collection $\{x_i\}\subset X$, $\|\sum \{K(I_i)\}(x_i)\|\leqq M \cdot \max_j\|\sum_{i=1}^{j} x_i\|$, then K is said to be convex-Gowurin, and the smallest such constant M shall be denoted by WK. Let T be a continuous linear operator from $C(X)$ into Y. Since

$$T(f) = T(f - \chi_{[0,1]}f(0)) + T(\chi_{[0,1]}f(0))$$

we shall consider T restricted to $C_\theta(X)$, the subset of functions $f\in C(X)$ satisfying $f(0)=\theta$.

THEOREM 1. *If K is a set function with values in $B[X, Y]$ which is convex-Gowurin and convex with respect to length* [1], *then $T(f)=v\int Kdf$ is a bounded linear operator from $C_\theta(X)$ to $\overline{Y}$, the completion of Y, and $\|T\|=WK$.*

PROOF. Suppose σ and σ' are partitions of $[0, 1]$ and suppose $f\in C_\theta(X)$. Then,

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$$\left\| \sum_{\sigma} \{K((t_i, t_{i+1}])\}(\Delta_i f) - \sum_{\sigma'} \{K((t_j, t_{j+1}])\}(\Delta_j f) \right\|$$
$$\leqq \left\| \sum_{\sigma * \sigma'} \{K((t_\nu, t_{\nu+1})\}(\Delta_\nu(pf_\sigma - pf_{\sigma'})) \right\|$$
$$\leqq WK \max_{\mu} \left\| \sum_{\nu=1}^{\mu} \Delta_\nu(pf_\sigma - pf_{\sigma'}) \right\|,$$

where pf_σ denotes the polygonal function determined by f and σ. As the mesh-fineness of σ and σ' tend to zero, the above tends to zero. Hence, $T(f) = v\int Kdf$ exists. That T is a bounded operator with $\|T\| \leqq WK$ follows by a similar argument and is omitted.

Let Ψ_I denote the fundamental function of the interval I [1]. That is, if $I = (a, b]$, then

$$\begin{aligned} \Psi_I &= 0 && \text{if } t \leqq a, \\ &= (t-a)/(b-a) && \text{if } a \leqq t \leqq b, \\ &= 1 && \text{if } t \geqq b. \end{aligned}$$

Then $\|T\| \geqq WK$, since for $\{I_i\}$ disjoint in $[0, 1]$ and $\{x_i\} \subset X$,

$$\left\| \sum \{K(I_i)\}(x_i) \right\| = \left\| v\int K\left(\sum \Psi_{I_i} x_i\right) \right\| = \left\| T\left(\sum \Psi_{I_i} x_i\right) \right\|$$
$$\leqq \|T\| \cdot \left\| \sum \Psi_{I_i} \cdot x_i \right\|_\infty \leqq \|T\| \max_j \left\| \sum_{i=1}^{j} x_i \right\|.$$

From the two preceding inequalities we conclude $\|T\| = WK$.

THEOREM 2. *Suppose T is a bounded linear operator from $C_0(X)$ into Y. Then there is a unique set function K with values in $B[X, Y]$ which is convex-Gowurin and convex with respect to length such that $T(f) = v\int Kdf$ for each $f \in C_0(X)$. Furthermore, $\|T\| = WK$.*

PROOF. Define $\mathfrak{J}$ from the continuous real valued functions on $[0, 1]$ into $B[X, Y]$ by $[\mathfrak{J}(f)](x) = T(f \cdot x)$. It follows as in [2] that $\|\mathfrak{J}\| \leqq \|T\|$. Define the set function K on the intervals in $[0, 1]$ by $K(I) = \mathfrak{J}(\Psi_I)$. That K is convex with respect to length follows from the linearity of $\mathfrak{J}$ and from the manner in which fundamental functions combine. Observe that, for disjoint half open intervals $\{I_i\}$ in $[0, 1]$ and $\{x_i\} \in X$,

$$\left\| \sum \{K(I_i)\}(x_i) \right\| = \left\| \sum \{\mathfrak{J}(\Psi_{I_i})\}(x_i) \right\| = \left\| T\left(\sum \Psi_{I_i} \cdot x_i\right) \right\|$$
$$\leqq \|T\| \cdot \left\| \sum \Psi_{I_i} x_i \right\|_\infty = \|T\| \max_j \left\| \sum_{i=1}^{j} x_i \right\|,$$

hence, K is convex-Gowurin. Suppose $f \in C_\theta(X)$, then

$$T(f) = \lim_\sigma T(pf_\sigma) = \lim_\sigma T\left(\sum_\sigma \Psi_{I_i} \Delta_i f \right) = \lim_\sigma \sum_\sigma \{K(I_i)\}(\Delta_i f)$$
$$= v \int K df.$$

The last equality follows from Theorem 1 as does the fact that $\|T\| = WK$. Since K determines T uniquely on polygonal functions, which are dense in $C(X)$, K is unique.

Hence, in the case Y is complete, the representation given here is a characterization of the linear tranformations. Such a characterization is not immediate from [**2**] and [**3**].

References

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