

A v -INTEGRAL REPRESENTATION FOR THE CONTINUOUS LINEAR OPERATORS ON SPACES OF CONTINUOUSLY DIFFERENTIABLE VECTOR-VALUED FUNCTIONS

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ABSTRACT. Suppose X and Y are linear normed spaces, and C_1 is the space of continuously differentiable functions from $[0, 1]$ into X . The authors give a representation theorem for the linear operators from C_1 into Y in terms of the v -integral operating on the function as opposed to the derivative of the function.

1. **Introduction.** Let C_1 denote the space of continuously differentiable functions on the interval $[0, 1]$ with values in a linear normed space X , and let C_1 be normed by $\|f\|_1 = \|f\|_\infty + \|f'\|_\infty$. In the special case that X is the real numbers R , we denote the function space by C_1R . The following characterization of the continuous linear functionals T on C_1R is given in [1, Chapter 14, B. 36 (a), p. 344]: $T(f) = \alpha f(0) + \int_0^1 f' d\mu$, where α is scalar and μ is a regular borel measure. Employing the results in [6], one obtains the following representation (but not a characterization unless Y is weakly complete) of the continuous linear operators from C_1 into a linear normed space Y : $T(f) = \Psi(f(0)) + \int dK \cdot f'$, where Ψ is an element of $B[X, Y]$, the continuous linear operators from X into Y , and K is a finitely additive operator valued set function from the half open intervals into $B[X, Y^+]$ which is weakly regular and of bounded semivariation (Y^+ denotes the weak sequential extension of Y as defined in [6, §2]).

Another representation (which gives a characterization under the less restrictive condition that Y is complete) of these linear operators follows from [5]: $T(f) = \Psi_1(f(0)) + \Psi_2(f'(0)) + v \int K \cdot df'$, where Ψ_1 and Ψ_2 are elements of $B[X, Y]$ and the v -integral is the generalized Stieltjes-type integral of a convex operator valued set function which is convex-Gowurin and defined from the half open intervals into $B[X, Y]$, as defined in [4]. Since each of the three mentioned representations use the fact that C_1 can be decomposed into $X \oplus C$ (using Lemma 2.3 of this paper), each shares the shortcoming that the integrals operate on f' rather than on f . In this paper we use the v -

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integral to give a representation which involves f directly, rather than requiring one to compute f' .

For convenience we let $C_{1,0}$ denote the subspace of functions f from C_1 such that $f(0) = 0_X$, the additive identity in X . Let P denote the polygonal functions which are 0_X at 0 , and observe that $\|\cdot\|_1$ can be extended to $C_{1,0} + P$ by

$$\|f + p\|_1 = \|f + p\|_\infty + \|f' + p'\|_{\text{ess}},$$

where $f \in C_{1,0}$, $p \in P$, and $\|\cdot\|_{\text{ess}}$ denotes the essential supremum norm. For each $f \in C_{1,0}$ and partition σ of $[0, 1]$, we define the polygonal function pf_σ to have corners precisely at the points t_i where $\sigma = \{t_i\}_{i=1}^n$ and such that $pf_\sigma(t_i) = f(t_i)$ for each i .

Let \mathcal{I} denote the collection of half open intervals $(a, b]$. A set function K from \mathcal{I} into $B[X, Y]$ is said to be convex with respect to length provided that $K(E) = \sum_{i=1}^n \lambda_i K(E_i)$ whenever $\bigcup_{i=1}^n E_i = E$ and $E_i \cap E_j = \emptyset$ for $i \neq j$, and where $\lambda_i = l(E_i)/l(E)$, that is, the ratio of the length of E_i to the length of E . Such a function is said to be 1-Gowurin if there exists a constant WK such that for each finite collection of disjoint intervals $\{E_i\}_{i=1}^n$ and collection $\{x_i\}_{i=1}^n$ of elements from X ,

$$(1) \quad \left\| \sum [K(E_i)](x_i) \right\|_Y \leq WK \left\{ \max_j \left\| \sum_{i=1}^j x_i \right\| + \max_j \|x_j\|/l(E_j) \right\}.$$

The general procedure for obtaining the representation theorem for T is as follows. First it is established that $\lim_{|\sigma|} pf_\sigma = f$ for each $f \in C_{1,0}$. Then the polygonal functions are decomposed into linear sums of fundamental functions. The fundamental functions are not in $C_{1,0}$ but are imbedded along with $C_{1,0}$ into $C_{1,0}^+$, the weak sequential extension of $C_{1,0}$. This is accomplished with a continuous linear operator θ from $C_{1,0} + P$ into $C_{1,0}^+$. Then, the linear transformation T is extended to T^+ on $C_{1,0}^+$ in the natural way. Finally T^+ is composed with the θ map and the action on the fundamental functions determines the convex set function K necessary to represent T in terms of v -integration of K with respect to f . Thus the procedure is entirely analogous to that described in [3] to obtain representation theorems in terms of more classical-type integration processes.

2. Preliminary results. The principal result in this section is the following theorem.

2.1. THEOREM. *Suppose f is an element of C_1 . Then $\{pf_\sigma\}$ converges to f in the 1-norm as $|\sigma|$ converges to zero.*

Before proving Theorem 2.1, we first establish the following results for the calculus of vector functions. The first lemma is fundamental to Theorem 2.1, and in its proof we use the sliding interval technique [2].

2.2. LEMMA. *If f is a differentiable X -valued function, and $f'(t) = \theta_X$ for all t , then f is a constant.*

PROOF. Under the above hypothesis assume that f is not constant. Then, there are numbers $a, b \in [0, 1]$ such that $(f(b) - f(a))/(b - a) \neq \theta_X$. Define $c = (b + a)/2$, then one of $\|(f(b) - f(c))/(b - c)\|$ and $\|(f(c) - f(a))/(c - a)\|$ must have norm at least as large as $\alpha = \|(f(b) - f(a))/(b - a)\|$. Let $[a_1, b_1]$ denote the interval on which this holds. Proceed inductively constructing a nested sequence of intervals $\{[a_n, b_n]\}$ such that, for each n , $(b_n - a_n) = \frac{1}{2}(b_{n+1} - a_{n+1})$ and $\|(f(b_{n+1}) - f(a_{n+1})) / (b_{n+1} - a_{n+1})\| \geq \alpha$. Let $x_0 = \bigcap_{n=1}^{\infty} [a_n, b_n]$. Then it follows that $f'(x_0) \neq \theta_X$ contrary to hypothesis. Therefore, we conclude that f must be constant, and the lemma is established.

2.3. LEMMA. *If f is a continuously differentiable X -valued function, then $f(t) = f(0) + \int_0^t f'(s) ds$.*

PROOF. It follows as in the classical setting that $g'(t) = f'(t)$ where $g(x) = \int_0^x f'(t) dt$. Therefore, $f'(t) - g'(t) = \theta_X$ for each t from which it follows that $f(t) - g(t)$ is identically a constant, which is given by $f(0) - g(0) = f(0)$.

2.4. LEMMA. *If f is a continuously differentiable X -valued function, then the difference functions*

$$f_{h_1 h_2}(t) = (f(t + h_1) - f(t - h_2)) / (h_1 + h_2), \quad h_1, h_2 > 0,$$

converge uniformly to f' as $h_1 + h_2 \rightarrow 0$.

PROOF. f' is uniformly continuous on $[0, 1]$; hence, given $\epsilon > 0$ there is a $\delta > 0$ such that if $|t - t'| < \delta$, then $\|f'(t) - f'(t')\| < \epsilon$. Choose $h_1 + h_2 < \delta$. Then,

$$\begin{aligned} \|f_{h_1 h_2}(t) - f'(t)\| &= \left\| (h_1 + h_2)^{-1} \int_{t-h_2}^{t+h_1} (f'(s) - f'(t)) ds \right\| \\ &\leq (h_1 + h_2)^{-1} (\epsilon(h_1 + h_2)) = \epsilon, \end{aligned}$$

from which the lemma follows.

PROOF OF 2.1. Suppose $\epsilon > 0$. Lemma 2.4 implies that there is a $\delta > 0$ such that if $h_1 + h_2 < \delta$, $h_1, h_2 > 0$, then $\|f_{h_1, h_2} - f'\|_{\infty} < \epsilon/2$. Suppose σ is a partition whose mesh-fineness, $|\sigma|$, is less than δ . Let $[t_i, t_{i+1}]$ denote one of the intervals defined by σ . Then $(pf_{\sigma})'$ on (t_i, t_{i+1}) is

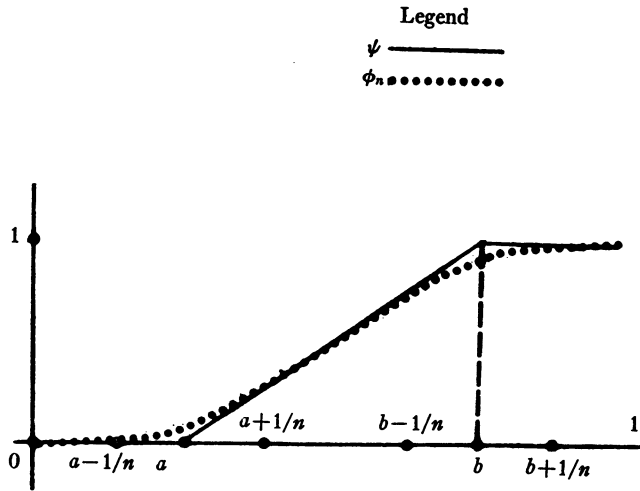


FIGURE 1a

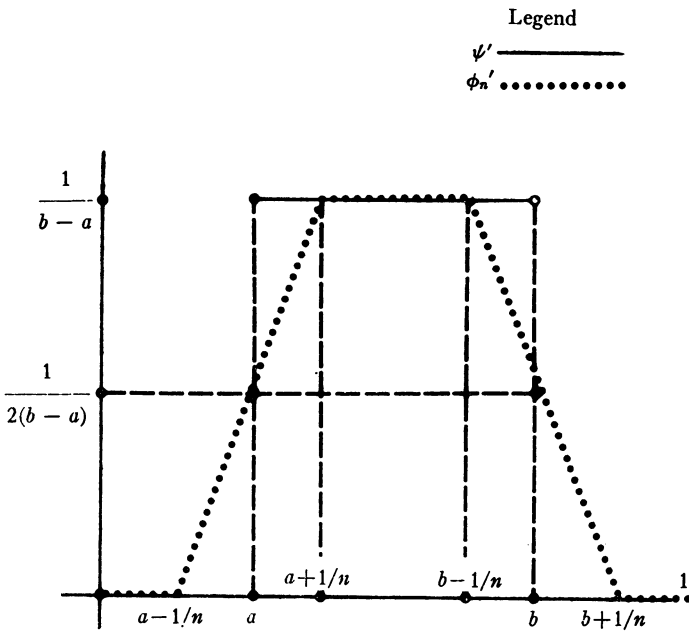


FIGURE 1b

given by $(f(t_{i+1}) - f(t_i)) / (t_{i+1} - t_i)$. Let $t \in (t_i, t_{i+1})$ and define $h_1 = t_{i+1} - t$ and $h_2 = t - t_i$. Then $(pf_\sigma)'(t) = f_{h_1, h_2}(t)$ and it follows that $\| (pf_\sigma)'(t) - f'(t) \| < \epsilon/2$ since $h_1 + h_2 = t_{i+1} - t_i < \delta$. Therefore,

$\|(\mathcal{P}f_\sigma)' - f'\|_{\text{ess}} < \epsilon/2$, from which we conclude $\|\mathcal{P}f_\sigma - f\|_\infty < \epsilon/2$ and, hence, that $\|\mathcal{P}f_\sigma - f\|_1 < \epsilon$. The theorem is proved.

3. The θ -map. In this section we construct a canonical, continuous linear mapping from $C_{1,0} + P$ into $C_{1,0}^+$. Suppose $(a, b) \in \mathcal{J}$. As in [4], let $\psi_{(a,b]}$ denote the fundamental function of the interval $(a, b]$ defined by $\psi_{(a,b]}(t) = 0$ if $0 \leq t \leq a$, $\psi_{(a,b]}(t) = (t-a)/(b-a)$ if $a \leq t \leq b$, and $\psi_{(a,b]}(t) = 1$ if $b \leq t \leq 1$. Define the sequence of continuous functions $\{\Phi_n\}$ by $\Phi_n \equiv 0$ if one of the following hold: $1/n \geq a$, $1/n \geq 1-b$, or $2/n \geq b-a$; and otherwise by

$$\begin{aligned} \Phi_n(x) &= 0 && \text{for } 0 \leq x \leq a - 1/n, \\ \Phi_n(x) &= [n/2(x-a+1/n)^2]/2(b-a) && \text{for } a - 1/n \leq x \leq a + 1/n, \\ \Phi_n(x) &= \psi_{(a,b]}(x) && \text{for } a + 1/n \leq x \leq b - 1/n, \\ \Phi_n(x) &= 1 - [1 - (n/2)(t-b+1/n)]^2/n(b-a) && \text{for } b - 1/n \leq x \leq b + 1/n, \end{aligned}$$

and

$$\Phi_n(x) = 1 \quad \text{for } b + 1/n \leq x \leq 1.$$

(See Figures 1a and 1b.)

Then $\{\Phi_n\}$ is a sequence in C_1R which is uniformly bounded and satisfies the properties that $\{\Phi_n\}$ converges pointwise to $\psi_{(a,b]}$, that $\{\Phi_n'\}$ converges pointwise to $\bar{\psi}_{(a,b]}$ which is defined by $\bar{\psi}_{(a,b]}(x) = \psi'(x)$ for $x \neq a$ or b and $\bar{\psi}(x) = 1/2(b-a)$ for $x = a$ or b . It follows from 13.36 [1, p. 345] that $\{\Phi_n\}$ is weakly convergent in C_1R . Hence, for $x \in X$ it follows that $\{\Phi_n \cdot x\}$ is weakly convergent in $C_{1,0}$. Let $\psi[(a, b], x]$ denote the weak limit of $\{\Phi_n \cdot x\}$ in C_1 . Define θ from $C_1 + P$ by $\theta(f + \sum_i \psi_{E_i} \cdot x_i) = f + \sum \psi[E_i, x_i]$ where $f \in C_1$ and $\sum_i \psi_{E_i} \cdot x_i \in P$.

3.1. LEMMA. *The linear map θ is continuous.*

In order to show θ is continuous it is sufficient to show that if $\{p_n\}$ is a sequence of polygonal functions converging to a function f in $C_{1,0}$ in the 1-norm, that $\{\theta(p_n)\}$ converges to f in $C_{1,0}^+$. A convenient tool to demonstrate such convergence is the following lemma which is established in [3].

3.2. LEMMA. *Suppose S is a linear normed space. A sufficient condition that a sequence $\{s_n^+\}$ in S^+ converges to $s_0^+ \in S^+$ is that, for each $n = 0, 1, \dots$, there exists a sequence $\{s_{n,m}\}_{m=1}^\infty$ in S such that*

- (a) *for each n , $\{s_{n,m}\}$ converges weakly to s_n^+ ;*
- (b) *$\|s_{n,m} - s_{0,m}\|$ converges to 0 as n, m increase without bound.*

PROOF OF 3.1. Suppose $\{p_n\} \subset P$ converges to $f \in C$ in the 1-norm. For each n , p_n has the form $\sum_{i=1}^{\nu(n)} \psi_{E_i^n} \cdot x_i^n$. Suppose ϵ is a positive number. For each pair (n, i) let $\{\Phi_{i,m}^n\}$ denote the sequence defined relative to $\psi_{E_i^n}$ as in this section. Then, $\{\sum_{i=1}^{\nu(n)} \Phi_{i,m}^n \cdot x_i^n\}$ converges weakly to $\theta(p_n)$ for each n . For each n define the sequence $\{h_{n,m}\}_{m=1}^\infty$ by $h_{n,m} = \sum_{i=1}^{\nu(n)} \Phi_{i,n(m)}^n \cdot x_i^n$ where $\{n(m)\}$ is a sequence of integers (depending on n) such that $\Phi_{i,n(m)}^n$ is not identically zero for $i=1, 2, \dots, \nu(n)$ and such that $n(\nu+1) = n(\nu) + 1$. It follows that for each n $\{h_{n,m}\}_{m=1}^\infty$ converges weakly to $\theta(p_n)$. Since $\{p_n\}$ converges to f in the 1-norm, there is an N such that $n \geq N$ and $\|p_n' - f'\|_{\text{ess}} < \epsilon/5$. Since f' is uniformly continuous, then there is a δ such that if $|x - x'| < \delta$, then $\|f'(x) - f'(x')\| < \epsilon/5$. Choose M such that $2/M < \delta$. Choose $n > N$ and $m > M$. Let $\{t_{i,n}\}_{i=1}^{\nu(n)}$ denote the domain values of the corner points of p_n . Then,

$$\|f'(t) - h'_{n,m}(t)\| = \|f'(t) - p'_n(t)\| < \epsilon/5,$$

for t in the complement of $\cup_{i=1}^{\nu(n)} [t_{i,n} - 1/n(m), t_{i,n} + 1/n(m)]$. Suppose $t \in (t_{j,n} - 1/n(m), t_{j,n} + 1/n(m))$. Then,

$$\begin{aligned} \|f'(t) - h'_{n,m}(t)\| &\leq \|f'(t) - f'(t_{j,n} - 1/n(m))\| \\ &\quad + \|f'(t_{j,n} - 1/n(m)) - p'_n(t_{j,n} - 1/n(m))\| \\ &\quad + \|p'_n(t_{j,n} - 1/n(m)) - h'_{n,m}(t)\|. \end{aligned}$$

However,

$$\begin{aligned} &\|p'_n(t_{j,n} - 1/n(m)) - h'_{n,m}(t)\| \\ &\leq \|p'_n(t_{j,n} - 1/n(m)) - p'_n(t_{j,n} + 1/n(m))\| \\ &\leq \|p'_n(t_{j,n} - 1/n(m)) - f'(t_{j,n} - 1/n(m))\| \\ &\quad + \|f'(t_{j,n} - 1/n(m)) - f'(t_{j,n} - 1/n(m))\| \\ &\quad + \|f'(t_{j,n} + 1/n(m)) - p'_n(t_{j,n} + 1/n(m))\| \\ &< \epsilon/5 + \epsilon/5 + \epsilon/5. \end{aligned}$$

It follows that $\|f'(t) - h'_{n,m}(t)\| < \epsilon/5 + \epsilon/5 + 3(\epsilon/5) = \epsilon$, and 2.3 implies that $\|f - h_{n,m}\| < \epsilon$. Therefore, 3.2 is satisfied, and it follows that $\{\theta(p_n)\}$ converges to f . Hence θ is continuous.

3.3. LEMMA. *The norm of the restriction of θ to P is 1.*

PROOF. It is sufficient to show that θ is norm-nonincreasing. Suppose $p = \sum \psi_{E_i} x_i \in P$. Then, $\{\sum \Phi_{i,m} x_i\}$ converges weakly to $\theta(p_n)$,

where for each i , $\{\Phi_{i,m}\}$ is the sequence constructed in this section. Then

$$\begin{aligned} \|\theta(p)\|_+ &= \sup \left\{ \lim_m \langle s^*, \sum \Phi_{i,m}x_i \rangle : s^* \in C_1^* \text{ and } \|s^*\| = 1 \right\} \\ &\leq \sup \left\{ \lim_m \sup \|s^*\| \left\| \sum \Phi_{i,m}x_i \right\|_1 : \|s^*\| = 1 \right\} \\ &< \lim_m \sup \left\| \sum_i \Phi_{i,m}x_i \right\|_\infty + \lim_m \sup \left\| \sum_i \Phi'_{i,m}x_i \right\|_\infty \\ &= \left\| \sum_i \psi_{E_i}x_i \right\|_\infty + \left\| \sum_i \psi'_{E_i}x_i \right\|_{\text{ess}} = \left\| \sum_i \psi_{E_i}x_i \right\|_1. \end{aligned}$$

4. The representation theorem.

4.1. THEOREM. *Suppose T is a continuous linear operator from C_1 into Y . Then there is a linear operator Φ from X into Y and a set function K from \mathcal{g} into $B[X, Y^+]$ which is convex with respect to length and which is 1-Gowurin such that $T(f) = \Phi(f(0)) + v \int Kdf$. Furthermore, the norm of T restricted to $C_{1,0}$ is equal to WK , where WK is the smallest constant satisfying inequality (1) from the Introduction of this paper.*

PROOF. We recall that the historical technique for obtaining an integral representation theorem for an operator T has been to extend the function space to include (usually in a weak sequential extension sense) the step functions or simple functions, which are dense in the extended norm on the extended function space. The action of the operator T on the step or simple functions determines an additive set function and finally approximation of a function by a linear combination of step or simple functions leads to an integral representation of the operator. Since we have continuously imbedded $C_{1,0} + P$ into $C_{1,0}^+$ with the θ map, we are now in a position to follow an analogous procedure where fundamental functions replace step or simple functions and the C_1 norm replaces sup norm. Consider T restricted to $C_{1,0}$. Then T has a natural extension from $C_{1,0}^+$ into Y^+ which we denote by T^+ . Define the set function K from \mathcal{g} into $B[X, Y^+]$ by $[K(E)](x) = T^+(\theta(\psi_E X))$. Then for $f \in C_{1,0}$ we have

$$\begin{aligned} T(f) &= T^+(\theta(f)) = T^+ \left(\theta \left(\lim_{|\sigma| \rightarrow 0} pf_\sigma \right) \right) = \lim_{|\sigma| \rightarrow 0} T^+(\theta(pf_\sigma)) \\ &= \lim_{|\sigma| \rightarrow 0} T^+(\theta(\sum \Delta_i f \psi_{E_i})) = \lim_{|\sigma| \rightarrow 0} \sum K(E_i) \Delta_i f = v \int Kdf. \end{aligned}$$

If the interval E is the union of the pairwise disjoint intervals $\{E_i\}_{i=1}^n$, then $\psi_E = \sum_{i=1}^n [l(E_i)/l(E)]\psi_{E_i}$ and consequently $K(E) = \sum_{i=1}^n \lambda_i K(E_i)$ and K is convex with respect to length. It is straightforward and follows as in Theorem 4.2 in [4] that K is 1-Gowurin and that $\|T+\theta\| \geq WK$. However, since θ is the identity map on $C_{1,0}$ and since the norm of θ on P is 1, we conclude that $\|T\| = WK$.

In general the converse of 4.1 does not hold unless Y is weakly sequentially complete. However, a partial converse does hold, and since its proof follows essentially along the lines of the proof of Theorem 4.2 in [4] we state the result without proof.

4.2. THEOREM. *Suppose K is a set function from \mathcal{g} with values in $B[X, Y]$. Then, $T(f) = \int Kdf$ is a continuous linear operator from $C_{1,0}$ into Y . Furthermore, $\|T\| = WK$.*

4.3. EXAMPLE. Suppose $x_0 \in (0, 1)$. The convex set function $K: \mathcal{g} \rightarrow \mathbf{R}$ defined by $K(a, b] = 0$ if $x_0 \in (0, a) \cup (b, 1)$, $K(a, b] = 1/(b-a)$ if $x_0 \in (a, b)$, and $K(a, b] = 1/2(b-a)$ if $x = a$ or if $x = b$, yields the transformation $T(f) = \int Kdf = f'(x_0)$.

4.4. EXAMPLE. The convex set function $K: \mathcal{g} \rightarrow B[\mathbf{R}, C^+] = C^+$ given by $[K(a, b)](x) = 0$ if $x \in (0, a) \cup (b, 1)$, $[K(a, b)](x) = 1/(b-a)$ if $x \in (a, b)$, and $[K(a, b)](x) = 1/2(b-a)$ if $x = a$ or if $x = b$ yields the transformation $T(f) = \int Kdf = f'$.

4.5. REMARK. Suppose f is differentiable n times. Then to obtain $f^{(n)}(x)$ one applies 4.4 ($n-1$) times and then 4.3.

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