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POLYNOMIAL EXTREMAL PROBLEMS IN L^p[1]

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ABSTRACT. For $p>2$, let $m_{p,n}$ be the minimum of the L^p norm all nth degree polynomials $\sum^n a_k e^{ikt}$ which satisfy $|a_k|=1$, $k=0, 1, \cdots, n$. We exhibit certain polynomials P_n whose L^p norm $(2<p<\infty)$ is asymptotic to $\sqrt{n}$, thereby proving that $m_{p,n}$ is itself asymptotic to $\sqrt{n}$. We also show that the sup norm of (essentially) the same polynomials is asymptotic to $(1.1716\ldots)\times\sqrt{n}$.

1. Introduction. Behind a number of polynomial extremal problems lies the following crude question: How close can we get to a situation where $P(z)$ is a polynomial of degree $n>0$, which, on the one hand, has coefficients of constant modulus, and on the other hand, $|P(z)|$ is constant for $|z|=1$?

Actually, for each $p>0$, one can formulate a precise L^p interpretation of the above question. Let $\mathcal{P}_n$ be the class of all nth degree polynomials $\sum_{k=0}^n a_k z^k$ such that $|a_k|=1$, $k=0, 1, \cdots, n$. For $0<p<\infty$, let

$$M_p(f) = \left((2\pi)^{-1}\int_{-\pi}^{\pi} |f(e^{i\theta})|^p d\theta\right)^{1/p}$$

so that for $f\in\mathcal{P}_n$, $M_2(f)=(\sum_{k=0}^n |a_k|^2)^{1/2}=(n+1)^{1/2}$. Let $M_\infty(f)=\sup_{\{\theta\}}|f(e^{i\theta})|$. From Hölder's inequality we can conclude that

$$(1) \qquad M_p(f) \leqq M_q(f) \qquad (0<p\leqq q\leqq\infty).$$

For $p>2$, the problem is to minimize $M_p(f)$. Let

$$m_{p,n} = \min_{\{f\}} M_p(f) \qquad (f\in\mathcal{P}_n).$$

By (1) we have $M_p(f)\geqq M_2(f)=(n+1)^{1/2}$, so that $m_{p,n}\geqq(n+1)^{1/2}$.

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Now, in order that $M_p(f)$ be [close to] $(n+1)^{1/2}$, the inequality $M_p(f) \geqq M_2(f)$—and therefore the underlying Hölder inequality—must be [close to] equality, i.e., $|f(e^{i\theta})|$ must be close to constant.

The main result of this paper is that, for $2<p<\infty$, $\mathfrak{m}_{p,n}$ is asymptotic to $\sqrt{n}$ as $n\to\infty$, so that, in this sense, our original question is answered. The problem of the minimum of the sup norm, i.e., $\mathfrak{m}_{\infty,n}$, is more elusive. It has been known for more than 50 years that $\mathfrak{m}_{\infty,n}$ satisfies $\mathfrak{m}_{\infty,n} \leqq c\sqrt{n}$, c an absolute constant (see Zygmund [7, Theorem 4.7, p. 199] and J. E. Littlewood [5, p. 27]). Littlewood [3], [4] showed that $\mathfrak{m}_{\infty,n} \leqq (1.35)\sqrt{n}$. But P. Erdös [2] conjectured that $\mathfrak{m}_{\infty,n}$ is not asymptotic to $\sqrt{n}$, and that, in fact, there exists an absolute constant $A>0$ such that $\mathfrak{m}_{\infty,n} \geqq (1+A)\sqrt{n}$. In this paper we will show that $\mathfrak{m}_{\infty,n} < (1.1717)\sqrt{n}$ by using polynomials similar to those used in the main result.

Before proceeding with $p>2$, let us see what happens when $0<p<2$. The inequality becomes reversed: $M_p(f) \leqq M_2(f) = (n+1)^{1/2}$, and the corresponding quantity to be considered is $\boldsymbol{M}_{p,n} = \max_{\{f\}} M_p(f)$ $(f\in \mathcal{P}_n)$. D. J. Newman [6] constructed polynomials P_n and proved the following lemma: $M_4^4(P_n) = n^2 + O(n^{3/2})$. He then used the lemma to prove that $\boldsymbol{M}_{1,n}/\sqrt{n}\to 1$, and, in fact, $\boldsymbol{M}_{1,n} \geqq \sqrt{n} - c$. By (1), the same follows immediately for $\boldsymbol{M}_{p,n}$, $1 \leqq p<2$. The result can be further extended to cover all p, $0<p<2$, as follows.

From the Schwarz inequality we conclude that

$$\int |f|^2 \leqq \left(\int |f|^{4-p}\right)^{1/2} \left(\int |f|^p\right)^{1/2}$$

which, applied to our present case, yields

$$M_p(P_n) \geqq \frac{(M_2(P_n))^{4/p}}{(M_{4-p}(P_n))^{(4-p)/p}} \geqq \frac{(n+1)^{2/p}}{(M_4(P_n))^{(4-p)/p}}$$

so that, applying the lemma, we obtain

$$M_p(P_n) \geqq \frac{(n+1)^{2/p}}{(n^2+An^{3/2})^{(4-p)/4p}} > \frac{n^{1/2}}{(1+An^{-1/2})^{(4-p)/4p}}$$
$$> n^{1/2}(1-(A/p)n^{-1/2}) > \sqrt{n} - A/p.$$

We thus record the more complete result corresponding to the theorem in [6]:

$\boldsymbol{M}_{p,n} \geqq \sqrt{n} - c/p$ $(0<p<2)$, where c is an absolute constant.

Yet another formulation of our original problem is as follows: Let $\mathfrak{F}_n$ be the class of all nth degree polynomials satisfying $\left|\sum a_k z^k\right| \leqq 1$ for $|z|=1$. We now consider $\mathfrak{M}_n = \max_{\{f\}} \sum |a_k|$ $(f\in\mathfrak{F}_n)$. Using the Schwarz inequality for sums, we have

$$\begin{aligned}\sum |a_k| &\leqq ((n+1)\sum |a_k|^2)^{1/2}\\ &= (n+1)^{1/2}M_2(\sum a_k z^k) \leqq (n+1)^{1/2}\end{aligned}$$

so that $\mathfrak{M}_n \leqq (n+1)^{1/2}$. If we are to have near equality in both of the above estimates, then both $|a_k|$ and $|\sum a_k z^k|$ must be nearly constant. Beller and Newman [1] have indeed shown that $\mathfrak{M}_n/\sqrt{n} \to 1$.

2. Main result.

THEOREM. $m_{p,n} \sim \sqrt{n}$, $2 < p < \infty$. *In fact, for sufficiently large* n, $(n+1)^{1/2} \leqq m_{p,n} \leqq \sqrt{n} + 2^{5p}(\log n)^{p-2}$.

REMARK. In all that follows, the phrase "for sufficiently large n" is to be understood. Its precise meaning is: for all $n \geqq K$, where K is some absolute constant (not depending on p or N).

PROOF. We use the same polynomials that Newman [6] constructed, namely

$$P_n(z) = \sum_{k=0}^{n} \exp(k^2\pi i/(n+1))z^k.$$

We will prove the following.

PROPOSITION 1. *For* $N = 1, 2, \cdots$,

$$M_{2^N}^{2^N}(P_n) \leqq n^{2^{N-1}} + (32)^{2^{N-1}} n^{(2^{N-1}-1/2)} (\log n)^{(2^{N-1}-2)}.$$

The theorem follows directly from Proposition 1. Indeed for $2^{N-1} \leqq p < 2^N$, we have

$$\begin{aligned}M_p(P_n) \leqq M_{2^N}(P_n) &\leqq \sqrt{n} + 2^{-N}(32)^{2^{N-1}}(\log n)^{(2^{N-1}-2)}\\ &< \sqrt{n} + 2^{5p}(\log n)^{p-2}.\end{aligned}$$

Before proving Proposition 1, we introduce the following notation. Given n, let $\{a_N(k)\}$ be defined by

$$(2) \qquad |P_n(e^{it})|^{2^N} = \sum_{k=-n2^{N-1}}^{n2^{N-1}} a_N(k)e^{ikt}, \qquad N = 1, 2, \cdots.$$

For $N \geqq 2$, the following relations are immediately evident:

$$(3) \qquad \begin{aligned} a_N(0) &= \sum_{k=-n2^{N-2}}^{n2^{N-2}} |a_{N-1}(k)|^2; \qquad a_N(j) = \overline{a_N(-j)},\\ a_N(j) &= \sum_{k=j-n2^{N-2}}^{n2^{N-2}} a_{N-1}(k)\overline{a_{N-1}(k-j)} \qquad (j > 0).\end{aligned}$$

We now define

$$(4)\qquad \begin{aligned} b_1(k) &= \sqrt{n} && (0 \leqq k \leqq \sqrt{n}),\\ &= n/k && (\sqrt{n} < k \leqq n/2) \end{aligned}$$

$$(5)\qquad b_1(n-k) = b_1(k) \quad (k \geqq 0); \qquad b_1(k) = b_1(-k).$$

For $N=2, 3, \cdots, b_N(k)$ is defined recursively:

$$(6)\qquad b_N(k) = (n \log n)^{2^{N-2}} b_{N-1}(k) \qquad (|k| \leqq n2^{N-2});$$

$$(7)\qquad \begin{aligned} &b_N(n2^{N-1}-k) = b_N(k) && (k \geqq 0),\\ &b_N(k) = b_N(-k); \quad b_N(k) = 0 && (|k| > n2^{N-1}). \end{aligned}$$

For $0 \leqq k \leqq n$, it follows from (6) that $b_N(k) = (n \log n)^{(2^{N-1}-1)} b_1(k)$, so that

$$(8)\qquad \begin{aligned} b_N(k) &= n^{(2^{N-1}-1/2)}(\log n)^{(2^{N-1}-1)} && (0 \leqq k \leqq \sqrt{n}),\\ &= (n^{2^{N-1}}/k)(\log n)^{(2^{N-1}-1)} && (\sqrt{n} < k \leqq n/2). \end{aligned}$$

Furthermore, it follows from (7) that

$$(9)\quad b_N(n2^m - k) = b_N(k) \qquad (0 \leqq k \leqq n2^{m-1};\ m = 0, 1, 2, \cdots, N-1).$$

We now state the following

LEMMA 1. *For* $N=1,2,\cdots$, $|a_N(k)| \leqq 2^{-5}3^{-N+1}(32)^{2^{N-1}} b_N(k)$ $(k=1, 2, \cdots, n2^{N-1})$.

PROOF OF PROPOSITION 1 AND LEMMA 1. Let $P(m)$ and $L(m)$ denote the truth of Proposition 1 and Lemma 1, respectively, for $N=m$. The proposition and lemma will be proved simultaneously by induction: $P(1)$ is trivial; $P(2)$ and $L(1)$ were proved by Newman [6]. Thus, it remains to be shown that $P(N-2)$, $P(N-1)$, and $L(N-2)$ together imply $L(N-1)$ and $P(N)$.

For $N \geqq 2$, by (2) and (3), we have

$$\begin{aligned} M_{2^N}^{2^N}(P_n) &= (1/2\pi)\int_{-\pi}^{\pi} \left(|P_n|^{2^{N-1}}\right)^2 d\theta\\ &= |a_{N-1}(0)|^2 + 2\sum_{k=1}^{n2^{N-1}} |a_{N-1}(k)|^2\\ &= \left[M_{2^{N-1}}^{2^{N-1}}(P_n)\right]^2 + 2\sum_{k=1}^{n2^{N-2}} |a_{N-1}(k)|^2. \end{aligned}$$

Applying $P(N-1)$, we obtain

$$
\begin{aligned}
M_{2^N}^{2^N}(P_n) \leqq\ & n^{2^{N-1}} + 2(32)^{2^{N-2}} n^{(2^{N-1}-1/2)} (\log n)^{(2^{N-2}-2)} \\
&+ (32)^{2^{N-1}} n^{(2^{N-1}-1)} (\log n)^{(2^{N-1}-4)} + 2\sum |a_{N-1}(k)|^2.
\end{aligned} \tag{10}
$$

Now, if we let $\sum'$ denote the summation excluding $j=0$ and $j=k$, then by (3), $L(N-2)$, $P(N-2)$, and (6), it follows that

$$
\begin{aligned}
|a_{N-1}(k)| &\leqq \sum_{j=k-n2^{N-3}}^{n2^{N-3}}{}' |a_{N-2}(j)a_{N-2}(j-k)| + 2|a_{N-2}(0)a_{N-2}(k)| \\
&\leqq 2^{-10}3^{-2N+6}(32)^{2^{N-2}} \sum_{j=k-n2^{N-3}}^{n2^{N-3}} b_{N-2}(j)b_{N-2}(j-k) \\
&\quad + \left(\frac{2^{-4}3^{-N+3}(32)^{2^{N-3}}}{(\log n)^{2^{N-3}}} + \frac{2^{-4}3^{-N+3}(32)^{2^{N-2}}}{(n^{1/2}\log^2 n)}\right) b_{N-1}(k).
\end{aligned}
$$

Set $c_N(k) = \sum_{j=k-n2^{N-2}}^{n2^{N-2}} b_{N-1}(j)b_{N-1}(j-k)$. We need the following inequality:

$$
c_N(k) \leqq (1.16)3^N b_N(k) \qquad (k = 0, 1, 2, \cdots, n2^{N-1};\ N = 2, 3, \cdots). \tag{11}
$$

Assuming (11), we have

$$
\begin{aligned}
|a_{N-1}(k)| &\leqq (1.16)2^{-10}3^{-N+5}(32)^{2^{N-2}} b_{N-1}(k) \\
&\quad \cdot \left(1 + \frac{7}{(32 \log n)^{2^{N-3}}} + \frac{7}{n^{1/2}\log^2 n}\right) \\
&\leqq 2^{-5}3^{-N+2}(32)^{2^{N-2}} b_{N-1}(k),
\end{aligned}
$$

i.e., $L(N-1)$ holds.

Proof of (11). We will first prove a preliminary fact, namely,

$$
c_N(k) \leqq (.58)3^N b_N(k) \quad (k = 0, 1, 2, \cdots, n2^{N-3};\ N = 2, 3, \cdots). \tag{12}
$$

We proceed by induction. First we show that (12) is true for $N=2$. By the Schwarz inequality, and (4), (5), we have

$$
\begin{aligned}
c_2(k) &\leqq \left(\sum_{j=k-n}^{n} b_1^2(j) \sum_{j=k-n}^{n} b_1^2(j-k)\right)^{1/2} = \sum_{j=k-n}^{n} b_1^2(j) \leqq \sum_{j=-n}^{n} b_1^2(j) \\
&\leqq 4 \sum_{0 \leqq j \leqq \sqrt{n}} n + 4 \sum_{\sqrt{n} < j \leqq n/2} n^2/j^2 \leqq 4n^{3/2} + 4n^2((\sqrt{n}-1)^{-1} - 2n^{-1}) \\
&\leqq 8n^{3/2}.
\end{aligned}
$$

Thus, for $0 \leqq k \leqq \sqrt{n}$, we have $c_2(k) \leqq 8n^{3/2} \leqq n^{3/2} \log n = b_2(k)$, while for $\sqrt{n} < k \leqq 2\sqrt{n}$, we have $c_2(k) \leqq 8n^{3/2} \leqq 16n^2/k \leqq (n^2 \log n)/k = b_2(k)$.

Before proceeding, let us note that

$$c_N(k) = 2\sum_{j=(k/2)+1/2}^{n2^{N-2}} b_{N-1}(j)b_{N-1}(j-k) \qquad (k \text{ odd}),$$

$$= b_{N-1}^2(k/2) + 2\sum_{j=(k/2)+1}^{n2^{N-2}} b_{N-1}(j)b_{N-1}(j-k) \qquad (k \text{ even}).$$

Now, for $2\sqrt{n} < k \leqq n/2$, if k is, say, odd, we have

$$\tfrac{1}{2}c_2(k) = \sum_{j=(k/2)+1/2}^{n} b_1(j)b_1(j-k) = n\sum_{(k/2)+1/2\leqq j\leqq n/2} j^{-1}b_1(j-k)$$
$$+ n\sum_{n/2<j<n-\sqrt{n}} (n-j)^{-1}b_1(j-k) + \sqrt{n}\sum_{n-\sqrt{n}\leqq j\leqq n} b_1(j-k).$$

We consider two cases separately:

(I) $2\sqrt{n} < k \leqq (n/2)-\sqrt{n}$. In this case,

$$\tfrac{1}{2}c_2(k) = n^2\sum_{(k/2)+1/2\leqq j<k-\sqrt{n}} j^{-1}(k-j)^{-1} + n^{3/2}\sum_{k-\sqrt{n}\leqq j\leqq k+\sqrt{n}} j^{-1}$$
$$+ n^2\sum_{k+\sqrt{n}<j\leqq n/2} j^{-1}(j-k)^{-1} + n^2\sum_{n/2<j\leqq k+(n/2)} (n-j)^{-1}(j-k)^{-1}$$
$$+ n^2\sum_{k+(n/2)<j<n-\sqrt{n}} (n-j)^{-1}(n-j+k)^{-1}$$
$$+ n^{3/2}\sum_{n-\sqrt{n}\leqq j\leqq n} (n-j+k)^{-1}.$$

Let us consider the first summation. Applying the inequality

$$\log(N/(M-1)) - \tfrac{1}{2}(M-1)^{-1}$$
$$\leqq \sum_{j=M}^{N}(1/j) \leqq \log(N/(M-1)) + \tfrac{1}{2}N^{-1},$$

we obtain

$$n^2\sum_{(k/2)+1/2\leqq j<k-\sqrt{n}} j^{-1}(k-j)^{-1}$$
$$= (n^2/k)\left(\sum_{(k/2)+1/2\leqq j<k-\sqrt{n}}(1/j) + \sum_{\sqrt{n}<j\leqq(k/2)-1/2}(1/j)\right)$$
$$\leqq (n^2/k)\left(\log\left(\frac{k-\sqrt{n}}{\sqrt{n}-2}\right) + \tfrac{1}{2}n^{-1/2} + \tfrac{1}{2}(\sqrt{n}-\tfrac{1}{2})^{-1}\right)$$
$$\leqq (n^2/k)(\log(\sqrt{n}/2) + (4/\sqrt{n})) \leqq (n^2/2k)\log n.$$

In a similar manner (making use also of the estimate $\log(1+x)<x$, $x>0$), one can find upper bounds on the other five summations, so that we end up with the estimate $\frac{1}{2}c_2(k) \leqq (2.6)(n^2/k)\log n$.

(II) $n/2-\sqrt{n}<k\leqq n/2$. In this case, $\frac{1}{2}c_2(k)$ breaks up into six summations which are a bit different from those in case (I). Here too, it can be verified that $\frac{1}{2}c_2(k) \leqq (2.6)(n^2/k)\log n$.

If k is even $(2\sqrt{n}<k\leqq n/2)$, then

$$c_2(k) = n^2/k^2 + \sum_{j=(k/2)+1}^{n} b_1(j)b_1(j-k),$$

and the same bound can again be gotten for $c_2(k)$. Thus, for $0\leqq k \leqq n/2$, $c_2(k)$ satisfies $c_2(k)\leqq (5.2)b_2(k)$, i.e., (12) holds for $N=2$.

Let us assume, now, that (12) holds for $N-1$. For $0\leqq k\leqq n2^{N-3}$, if k is, say, odd, then by (6) and (7) we have

$$\begin{aligned}
\frac{c_N(k)}{(n\log n)^{2N-2}} &= \frac{2}{(n\log n)^{2N-2}} \sum_{j=(k/2)+1/2}^{n2^{N-2}} b_{N-1}(j)b_{N-1}(j-k) \\
&= 2\sum_{j=(k/2)+1/2}^{n2^{N-3}} b_{N-2}(j)b_{N-2}(j-k) \\
&\quad + 2\sum_{j=n2^{N-3}+1}^{k+n2^{N-3}} b_{N-2}(n2^{N-2}-j)b_{N-2}(j-k) \\
&\quad + 2\sum_{j=k+n2^{N-3}+1}^{n2^{N-2}} b_{N-2}(n2^{N-2}-j)b_{N-2}(n2^{N-2}-j+k) \\
&= 2\left\{\sum_{j=(k/2)+1/2}^{n2^{N-3}} + \sum_{j=1}^{n2^{N-3}}\right\} b_{N-2}(j)b_{N-2}(j-k) \leqq 3c_{N-1}(k) \\
&\leqq 3(.58)3^{N-1}b_{N-1}(k) \\
&= \frac{(.58)3^N b_N(k)}{(n\log n)^{2N-2}}\cdot
\end{aligned}$$

The same can be seen to hold if k is even. Thus, $c_N(k)\leqq(.58)3^N b_N(k)$, which proves (12).

We now prove a somewhat weaker form of (11), namely

$$c_N(k) \leqq (1.16)3^N b_N(k) \qquad \text{for } 0\leqq k\leqq n2^{N-2}. \tag{13}$$

Let $0\leqq k\leqq n2^{N-3}$. Then

$$
\begin{aligned}
c_N(n2^{N-2}-k) &= \sum_{j=-k}^{n2^{N-2}} b_{N-1}(j)b_{N-1}(j+k-n2^{N-2}) \\
&= \sum_{j=-k}^{n2^{N-2}-k} b_{N-1}(j)b_{N-1}(j+k) \\
&\quad + \sum_{j=n2^{N-2}-k+1}^{n2^{N-2}} b_{N-1}(j)b_{N-1}(j+k-n2^{N-2}) \\
&= \left\{ \sum_{j=k-n2^{N-2}}^{k} + \sum_{j=0}^{k-1} \right\} b_{N-1}(j)b_{N-1}(j-k) \leqq 2c_N(k) \\
&\leqq (1.16)3^N b_N(k) = (1.16)3^N b_N(n2^{N-2}-k)
\end{aligned}
$$

by (12), thus proving (13).

Finally, for $0 \leqq k \leqq n2^{N-2}$, we have

$$
\begin{aligned}
c_N(n2^{N-1}-k) &= \sum_{j=n2^{N-2}-k}^{n2^{N-2}} b_{N-1}(j)b_{N-1}(j+k-n2^{N-1}) \\
&= \sum_{j=-k}^{0} b_{N-1}(j+n2^{N-2})b_{N-1}(j+k-n2^{N-2}) \\
&= \sum_{j=0}^{k} b_{N-1}(j)b_{N-1}(j-k) \\
&\leqq c_N(k) \leqq (1.16)3^N b_N(k) = (1.16)3^N b_N(n2^{N-1}-k)
\end{aligned}
$$

by (13), which proves (11).

Returning to the proof of Proposition 1, we note that it follows from (9) that

$$
\sum_{k=1}^{n2^{N-2}} b_{N-1}^2(k) = 2^{N-1}\sum_{k=1}^{n/2} b_{N-1}^2(k).
$$

Thus, applying $L(N-1)$, we obtain

$$
\begin{aligned}
2\sum_{k=1}^{n2^{N-2}} \left| a_{N-1}(k) \right|^2 &\leqq 2^{-9}3^{-2N+4}(32)^{2^{N-1}} \sum_{k=1}^{n2^{N-2}} b_{N-1}^2(k) \\
&= 2^{N-10}9^{-N+2}(32)^{2^{N-1}} \sum_{k=1}^{n/2} b_{N-1}^2(k).
\end{aligned}
$$

Combining this with (8), we get

$$2\sum_{k=1}^{n2^{N-2}} |a_{N-1}(k)|^2 \leqq 2^{-8}(2/9)^{N-2}(32)^{2^{N-1}}$$

$$\cdot\left\{\sum_{1\leq k\leq\sqrt{n}} n^{(2^{N-1}-1)}(\log n)^{(2^{N-1}-2)} + \sum_{\sqrt{n}<k\leq n/2} (n^{2^{N-1}}/k^2)(\log n)^{(2^{N-1}-2)}\right\}$$

$$\leqq 2^{-7}(2/9)^{N-2}(32)^{2^{N-1}} n^{(2^{N-1}-1/2)}(\log n)^{(2^{N-1}-2)}.$$

Combining this with (10), we obtain

$$M_{2^N}^{2^N}(P_n) \leqq n^{2^{N-1}} + 2^{-7}(32)^{2^{N-1}} n^{(2^{N-1}-1/2)}(\log n)^{(2^{N-1}-2)}$$

$$\cdot\left\{(2/9)^{N-2} + \frac{2^8}{(32\log n)^{2^{N-2}}} + \frac{2^7}{n^{1/2}\log^2 n}\right\}$$

$$\leqq n^{2^{N-1}} + (32)^{2^{N-1}} n^{(2^{N-1}-1/2)}(\log n)^{(2^{N-1}-2)},$$

which proves Proposition 1 (and Lemma 1).

3. An upper bound for the minimum of the sup norm.

PROPOSITION 2. *$m_{\infty,n} < (1.1717)\sqrt{n}$ for sufficiently large n.*

PROOF. Let $\mathfrak{F}(x, y) = \int_x^y \exp(\frac{1}{2}\pi u^2 i)du$, so that $\mathfrak{F}(0, x)$ is the familiar Fresnel integral. Let $A > 0$ be the value of x for which $|\mathfrak{F}(-\infty, x)|$ is a maximum, and let M be that maximum: $M = 1.6566\cdots$, so that $M/\sqrt{2} = 1.1716\cdots$. Proposition 2 follows directly from the following

LEMMA 2. *Let*

$$P_n(z) = \sum_{k=0}^{n} \exp\left(\frac{k^2\pi i}{a_n(n+1)}\right)z^k,$$

where $a_n = 1 + (n+1)^{-1/4}$. Then

$$\max_{\{\theta\}} |P_n(e^{i\theta})| = (M/\sqrt{2})\sqrt{n} + O(n^{1/4}).$$

PROOF OF LEMMA 2. Let

$$F_\theta(u) = \exp\left(\frac{u^2\pi i}{a_n(n+1)} + i\theta u\right)$$

so that $P_n(e^{i\theta}) = \sum_{k=0}^{n} F_\theta(k)$. Let $f_\theta(u) = u^2/(2a_n(n+1)) + u\theta/(2\pi)$, so that $F_\theta(u) = e^{2\pi i f_\theta(u)}$. Now for θ satisfying

$$(14)\qquad -\pi - \pi/a_n \leqq \theta < \pi - \pi/a_n$$

we have $|f'_\theta(u)| \leqq 1-\frac{1}{2}(1-1/a_n)$ for u in the interval $[0, n+1]$.

REMARK. From now on, it is understood that θ satisfies (14).

Since, furthermore, $f'_\theta(u)$ is monotone, we can apply the following lemma of van der Corput, which we state in the notation of Zygmund [7, p. 198], although in somewhat greater generality.

LEMMA 3. *If $f'(u)$ is monotone and $|f'| \leqq 1-\epsilon$ in (a, b) $(0<\epsilon<1)$, then $|D(F; a, b)| \leqq A/\epsilon$, where A is an absolute constant.*

(Zygmund proves it for the case $\epsilon = \frac{1}{2}$, by showing that D numerically does not exceed $1+(2/\pi)\sum_{n=1}^{\infty} n^{-1}(n-\frac{1}{2})^{-1}$. It is not hard to see that for any ϵ, the bounding series becomes $(4/\pi)\sum_{n=1}^{\infty} n^{-1}(n-1+\epsilon)^{-1}$. Since the first term of the series is $1/\epsilon$ and the sum of the remaining terms is less than 1, Lemma 3 follows immediately.)

In our case, Lemma 3 yields

$$\left| \int_0^{n+1} F_\theta(u)du - \sum_{k=0}^{n} F_\theta(k) \right| \leqq \frac{2A}{1 - 1/a_n} < 4An^{1/4}. \tag{15}$$

Now, by making the change of variables $v = (2/a_n)^{1/2}(n+1)^{-1/2}u + (\theta/\pi)(\frac{1}{2}a_n(n+1))^{1/2}$, we have

$$\int_0^{n+1} F_\theta(u)du = (\tfrac{1}{2}a_n(n+1))^{1/2}e^{ia}\mathfrak{F}(t, (2/a_n)^{1/2}(n+1)^{1/2}+t), \tag{16}$$

where $t = (\theta/2\pi)(2a_n(n+1))^{1/2}$ and a is real.

We now show that

$$\max_{\{t\}} \left| \mathfrak{F}(t, (2/a_n)^{1/2}(n+1)^{1/2}+t) \right| = M + O(n^{-1/2}). \tag{17}$$

In accordance with the Remark, the maximum is taken over all t satisfying

$$-2^{-1/2}(n+1)^{1/2}(a_n^{1/2} + a_n^{-1/2}) \leqq t < 2^{1/2}(n+1)^{1/2}(a_n^{1/2} - a_n^{-1/2}). \tag{18}$$

Before proving (17), let us note that by making the change of of variables $v = u^2$ and integrating by parts, we get

$$\mathfrak{F}(x, \infty) = (\pi x)^{-1}i\cdot\exp(\tfrac{1}{2}\pi x^2 i) + O(x^{-3}),$$

so that for sufficiently large x,

$$|\mathfrak{F}(x, \infty)| < (3x)^{-1}. \tag{19}$$

Now let t_n be a value of t for which $|\mathfrak{F}(t, (2/a_n)^{1/2}(n+1)^{1/2}+t)|$

attains its maximum. Since $\mathfrak{F}(x, y)=\mathfrak{F}(-y, -x)$, we may assume that $t_n \leqq -(2/a_n)^{1/2}(n+1)^{1/2}-t_n$, i.e.,

$$t_n \leqq -(2a_n)^{-1/2}(n+1)^{1/2}. \tag{20}$$

If n is sufficiently large so that $-A$ is greater than the lower bound in (18), then by (19) we have, on the one hand,

$$\begin{aligned} &\left|\mathfrak{F}(t_n, (2/a_n)^{1/2}(n+1)^{1/2}+t_n)\right| \\ &\qquad \geqq \left|\mathfrak{F}(-A, (2/a_n)^{1/2}(n+1)^{1/2}-A)\right| \\ &\qquad \geqq \left|\mathfrak{F}(-A, \infty)\right| - \left|\mathfrak{F}((2/a_n)^{1/2}(n+1)^{1/2}-A, \infty)\right| \\ &\qquad \geqq M - \frac{1}{3((2/a_n)^{1/2}(n+1)^{1/2}-A)} \geqq M - \tfrac{1}{3}n^{-1/2}, \end{aligned}$$

and on the other hand, by (19) and (20),

$$\begin{aligned} &\left|\mathfrak{F}(t_n, (2/a_n)^{1/2}(n+1)^{1/2}+t_n)\right| \\ &\qquad \leqq \left|\mathfrak{F}(-\infty, (2/a_n)^{1/2}(n+1)^{1/2}+t_n)\right| + \left|\mathfrak{F}(-\infty, t_n)\right| \\ &\qquad \leqq \left|\mathfrak{F}(-\infty, A)\right| + \tfrac{1}{3}\left|t_n\right|^{-1} \leqq M + \tfrac{1}{2}n^{-1/2} \end{aligned}$$

for sufficiently large n, which proves (17).

Combining (15), (16), and (17), we obtain

$$\begin{aligned} \max_{\{\theta\}}\left|P_n(e^{i\theta})\right| &= 2^{-1/2}(1+(n+1)^{-1/4})^{1/2}(n+1)^{1/2}M + O(n^{1/4}) \\ &= (M/\sqrt{2})\sqrt{n} + O(n^{1/4}). \end{aligned}$$

Q.E.D.

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