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ON NONLINEAR EQUATIONS OF HAMMERSTEIN TYPE IN BANACH SPACES[1]

PETER HESS

ABSTRACT. A new theorem on the existence and uniqueness of a solution of an equation of Hammerstein type $u+TNu=f$ is given. Here N denotes a (nonlinear) monotone mapping of a real reflexive Banach space X into its conjugate space X^* and T a bounded monotone linear operator of X^* into X. It is not assumed that T or N is coercive.

In operator-theoretic terms, a nonlinear Hammerstein integral equation

$$u(x) + \int_G K(x, y)a(y, u(y))dy = f(x)$$

can be written as a functional equation

$$u + TNu = f, \tag{1}$$

defined in a Banach space X of functions on G, and with the linear and nonlinear mappings T, N being given by

$$(Tu)(x) = \int_G K(x, y)u(y)dy; \qquad (Nu)(x) = a(x, u(x)).$$

In this note we present a new theorem on the existence and uniqueness of a solution of equation (1), sharpening results by Amann [**1**, Theorem 1] and Browder-Gupta [**5**, Theorem 2]. We employ the following definitions: If X is a real Banach space, X^* its conjugate space, we let (w, u) denote the duality pairing between elements $w \in X^*$ and $u \in X$. A subset M of the product space $X \times X^*$ is said to be *monotone* if $(w_1 - w_2, u_1 - u_2) \geqq 0$ for all $[u_1, w_1] \in M$, $[u_2, w_2] \in M$. The monotone set M is *maximal monotone* if it is not properly contained in any other monotone set. For a mapping A of X into the set 2^{X^*} of all subsets of X^*, the graph $G(A)$ of A is the subset of $X \times X^*$ given by $G(A) = \{[u, w] \in X \times X^* : w \in Au\}$, and A is said to be

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monotone (maximal monotone) if its graph $G(A)$ is a monotone (maximal monotone) set. The mapping $A: X \to 2^{X^*}$ is further *coercive* if there exists a continuous function c from R^+ into R^1 with $c(r) \to +\infty$ as $r \to +\infty$, such that $(w, u) \geqq c(\|u\|)\|u\|$ for all $[u, w] \in G(A)$. A (single-valued) mapping of X into X^* is finally *hemicontinuous* if it is continuous from each line segment in X to the weak topology on X^*.

THEOREM. *Let N be a hemicontinuous monotone mapping of the real reflexive Banach space X into X^*, and let the monotone linear operator $T: X^* \to X$ satisfy the following condition:*

(*) *There exists a constant $d > 0$ such that $(v, Tv) \geqq d\|Tv\|^2$ for all $v \in X^*$.*

Then the equation $u + TNu = f$ admits a unique solution for each f in X.

It is immediate to see that condition (*) implies the boundedness of the operator T. The mapping T is further maximal monotone, and as a consequence of condition (*), the maximal monotone inverse operator $T^{-1}: X \to 2^{X^*}$, defined by $G(T^{-1}) = \{[u, w] \in X \times X^* : [w, u] \in G(T)\}$ is coercive. In order to compare our Theorem with some of the existing results, we introduce the concept of *angle-boundedness*:

DEFINITION. *A bounded monotone linear mapping T of X^* into X is said to be angle-bounded with constant $\gamma \geqq 0$ if, for all v, w in X^*,*

$$|(v, Tw) - (w, Tv)| \leqq 2\gamma\{(v, Tv)\}^{1/2}\{(w, Tw)\}^{1/2}. \tag{2}$$

The orientation of our condition (*) with respect to angle-boundedness is given in the

PROPOSITION. *Let the real Banach space X be reflexive. Then any angle-bounded operator $T: X^* \to X$ satisfies condition* (*).

On the other hand, the following Example shows that condition (*) is a proper weakening of the assumption of angle-boundedness.

EXAMPLE. *Let H be a real separable (infinite-dimensional) Hilbert space with orthonormal basis $\{e_n\}$, $n = \pm 1, \pm 2, \cdots$, and let T be the bounded linear operator defined on this basis by $Te_n = n^{-2}e_n + n^{-1}e_{-n}$. Then T satisfies condition (*): $(v, Tv) \geqq \frac{1}{4}\|Tv\|^2$ for all $v \in H$, but T is not angle-bounded (as may be seen by introducing e_n and e_{-n} as v and w in the definition (2) and letting $|n| \to \infty$).*

In the special case where the Banach space X has a certain imbedding property and T is angle-bounded, the assertion of the Theorem has been proved by Amann [1]. Browder and Gupta have shown in [5] that the additional imbedding assumption on X is unnecessary. We remark that in contrary to our Theorem which demands the reflexivity of the space X, the results in [1] and [5] are formulated for equation (1) in the conjugate space X^* of some (not necessarily reflexive) Banach space X. The writer observed in [7] that with the method of [5], equation (1) can also be discussed in an arbitrary Banach space X.

Applications of our Theorem to more general Hammerstein equations are given in [8], where we derive existence theorems of Fredholm alternative type for asymptotically homogeneous and odd operators.

PROOF OF THE THEOREM. We first show the *existence* of a solution. Equation (1) is equivalent to the relation

$$0 \in T^{-1}(u - f) + Nu$$

(cf. [4]). Let $v = u - f$, and let N_f denote the monotone mapping defined for each $v \in X$ by $N_f v = N(v + f)$. Then equation (1) holds if and only if

$$0 \in (T^{-1} + N_f)v. \tag{3}$$

By a result of Rockafellar [9] (see also [2], [6]), the mapping $T^{-1} + N_f$ is maximal monotone. Moreover, $T^{-1} + N_f$ is coercive. The solvability of equation (3) thus follows from a basic result on coercive maximal monotone mappings (e.g. [3]).

For the proof of the *uniqueness* of a solution of (1), we let u, v denote elements in X such that $u + TNu = f$, $v + TNv = f$. Then

$$(Nu - Nv, u - v) + (Nu - Nv, T(Nu - Nv)) = 0,$$

and by monotonicity of N and condition (*),

$$0 \geqq (Nu - Nv, T(Nu - Nv)) \geqq d\|T(Nu - Nv)\|^2.$$

It follows that $TNu = TNv$ and hence $u = v$. Q.E.D.

PROOF OF THE PROPOSITION. Let the bounded monotone linear operator T be angle-bounded. By a result of Browder-Gupta [5], there exist a Hilbert space H (whose norm and inner product we denote by $\|\cdot\|_H$ and $(\cdot, \cdot)_H$, respectively), a continuous linear mapping S of X^* into H, and a bounded linear monotone bijective mapping C of H onto itself such that $T = S^*CS$ and $(C^{-1}v, v)_H \geqq e\|v\|_H^2$ for all $v \in H$, with $e > 0$. For $u \in X^*$, we then have

$$(u, Tu) = (u, S^*CSu) = (Su, CSu)_H \geqq e\|CSu\|_H^2 \geqq e\|S\|^{-2}\|Tu\|^2. \text{ Q.E.D.}$$

REMARK. In recent years, various results on the solvability of Hammerstein equations involving *unbounded* maximal monotone linear operators T and coercive mappings N have been derived (e.g.

[4]). Since the concept of angle-boundedness can be defined also for unbounded monotone linear operators, the question is motivated whether one can obtain existence results (analogous to our Theorem) also for unbounded maximal monotone angle-bounded operators T and monotone, but noncoercive N. The following Example gives a negative answer.

EXAMPLE. *Let H be a real separable (infinite-dimensional) Hilbert space with orthonormal basis $\{e_n\}$, $n=\pm 1, \pm 2, \cdots$, let N be the bounded linear monotone operator defined on this basis by*

$$\begin{aligned} Ne_n &= e_{-n}, & n &\geqq 1, \\ &= -e_{-n}, & n &\leqq -1, \end{aligned}$$

and let T be the positive selfadjoint operator in H satisfying

$$\begin{aligned} Te_n &= n^2 e_n, & n &\geqq 1, \\ &= n^{-2} e_n, & n &\leqq -1. \end{aligned}$$

Then the equation $u+TNu=f$ is not solvable in H for $f=\sum_{n\geqq 1} n^{-1}e_{-n}$.

[2]ADDED IN PROOF. The observation that condition (*) is (even in the case of compact operators T) a proper weakening of the concept of angle-boundedness, permits to extend some recent results by Amann on Hammerstein equations with compact angle-bounded kernel T and bounded, continuous N. (*Hammersteinsche Gleichungen mit kompakten Kernen*, Math. Ann. **186** (1970), 334–340, Theorem 1; *Existence theorems for equations of Hammerstein type* (to appear in Applicable Analysis), Theorem 3). In addition, our proof is considerably simpler than those given by Amann.

THEOREM. *Let T be a compact linear operator of the real Banach space X into X^* satisfying condition*

(*) *There exists $d>0$ such that $(Tu, u)\geqq d\|Tu\|^2$ for all $u\in X$.*

Let further N be a bounded continuous mapping of X^* into X, and assume that for some function φ: $R^+\to R^+$ satisfying $\varphi(r)=o(r^2)$ as $r\to+\infty$, we have $(v, Nv)\geqq -c\|v\|^2-\varphi(\|v\|)$ for $v\in X^*$, with $c<d$. Then the equation $w+TNw=0$ admits a solution w in X^*.

PROOF. By the boundedness and continuity of the mapping N and the compactness of T, the operator TN: $X^*\to X^*$ is continuous and compact. We define a continuous mapping $C_t w=C(w, t)$:

[2] This "Added in proof" was printed erroneously with a previous article (*A variational approach to a class of nonlinear eigenvalue problems*, Proc. Amer. Math. Soc. **29** (1971), 272–276) by the author.

$X^* \times [0, 1] \to X^*$, by $C_t w = w + tTNw$. We first show that $C_t w \neq 0$ for $\|w\| = R$ sufficiently large and $t \in [0, 1]$. Indeed, $C_0 w \neq 0$ for $w \neq 0$. Let thus $0 < t \leqq 1$, and suppose $w + tTNw = 0$ for some $w \in X^*$. Then $0 = (w, Nw) + t(TNw, Nw) \geqq -c\|w\|^2 - \varphi(\|w\|) + td\|TNw\|^2$. Since $\|TNw\| = t^{-1}\|w\|$, we obtain $\varphi(\|w\|) + c\|w\|^2 \geqq t(\varphi(\|w\|) + c\|w\|^2) \geqq d\|w\|^2$, which implies the uniform boundedness of $\|w\|$. Next we observe that the degree theory of Leray-Schauder is applicable to the homotopy $\{C_t\}_{0 \leq t \leq 1}$. Since the degree $\deg(C_0, B_R, 0)$ of the mapping $C_0 = I$ on the open ball B_R in X^* with respect to the origin is $+1$ and this degree remains invariant through the homotopy $\{C_t\}_{0 \leq t \leq 1}$, we infer that the equation $C_1 w = w + TNw = 0$ admits a solution $w \in B_R$. Q.E.D.

References

1. H. Amann, *Ein Existenz- und Eindeutigkeitssatz für die Hammersteinsche Gleichung in Banachräumen*, Math. Z. **111** (1969), 175–190. MR **40** #7894.

2. H. Brezis, M. G. Crandall and A. Pazy, *Perturbations of nonlinear maximal monotone sets in Banach spaces*, Comm. Pure Appl. Math. **23** (1970), 123–144. MR **41** #2454.

3. F. E. Browder, *Nonlinear operators and nonlinear equations of evolution in Banach spaces*, Proc. Sympos. Pure Math., vol. 18, part 2, Amer. Math. Soc., Providence, R. I. (to appear).

4. F. E. Browder, D. G. deFigueiredo and C. P. Gupta, *Maximal monotone operators and nonlinear integral equations of Hammerstein type*, Bull. Amer. Math. Soc. **76** (1970), 700–705.

5. F. E. Browder and C. P. Gupta, *Monotone operators and nonlinear integral equations of Hammerstein type*, Bull. Amer. Math. Soc. **75** (1969), 1347–1353.

6. F. E. Browder and P. Hess, *Nonlinear mappings of monotone type in Banach spaces* (to appear).

7. P. Hess, *Nonlinear functional equations and eigenvalue problems in nonseparable Banach spaces*, Comment. Math. Helv. (to appear).

8. ———, *On nonlinear mappings of monotone type homotopic to odd operators*, J. Functional Analysis (to appear).

9. R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc. **149** (1970), 75–88.

University of Chicago, Chicago, Illinois 60637