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ON A GEOMETRIC PROPERTY OF THE SET OF INVARIANT MEANS ON A GROUP

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ABSTRACT. If G is a discrete group and $x \in G$ then $x^{\sim}$ denotes the homeomorphism of βG onto βG induced by left multiplication by x. A subset K of βG is said to be *invariant* if it is closed, nonempty and $x^{\sim}K \subset K$ for each $x \in G$. Let $ML(G)$ denote the set of left invariant means on G. (They can be considered as measures on βG.)

THEOREM. *Let G be a countably infinite amenable group and let K be an invariant subset of βG. Then the nonempty w^*-compact convex set $M(G, K) = \{\phi \in ML(G)$: suppt $\phi \subset K\}$ has no exposed points (with respect to w^*-topology). Therefore, it is infinite dimensional.*

1. Let S be a semigroup, $m(S)$ the Banach space of bounded real functions on S with the sup norm $\phi \in m(S)^*$ is called a mean if $\|\phi\| = 1$ and $\phi(f) \geqq 0$ for $f \geqq 0$. Let βS denote the Stone-Čech compactification of the discrete set S. Each $f \in m(S)$ can be extended to a continuous function on βS. The extended function will again be denoted by f. If ϕ is a mean on $m(S)$ then μ_ϕ will denote the probability measure on βS defined by $\int_{\beta S} f d\mu_\phi = \phi(f), f \in m(S)$.

A mean $\phi \in m(S)^*$ is said to be left invariant if $\phi(f) = \phi(l_s f)$ for $f \in m(S)$ and $s \in S$ where $l_s f \in m(S)$ is defined by $(l_s f)(s_1) = f(ss_1)$. Denote the set of all left invariant means on S by $ML(S)$. If $ML(S)$ is nonempty then we say S is left amenable. In this case, $ML(S)$ is w^*-compact convex (cf. [5]).

For $s \in S$, $s^{\sim}$ denotes the continuous mapping of βS into itself defined by $s^{\sim}s_1 = ss_1$, $s_1 \in S$. A subset K of βS is said to be *invariant* if it is nonempty, closed and $s^{\sim}K \subset K$ for each $s \in S$. If K is an invariant subset of βS, set $M(S, K) = \{\phi \in ML(S) : \text{suppt}\, \mu_\phi \subset K\}$.

If S is left amenable and K is an invariant subset of βS then, by Day's fixed point theorem [6], $M(S, K)$ is nonempty. It is also easy to check that $M(S, K)$ is w^*-compact convex and each extreme point of $M(S, K)$ is also an extreme point of $ML(S)$. Thus, by the Kreĭn-Milman theorem, $M(S, K)$ contains at least one extreme point of $ML(S)$. In general βS contains many mutually disjoint invariant sets. For example, when S is an infinite amenable group then βS

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has at least 2^c such sets (cf. [2]). Here c denotes the cardinality of the continuum. Therefore, we concluded in [2] that for each infinite amenable group S the set $ML(S)$ has at least 2^c extreme points. It is then natural to ask whether $ML(S)$ has any exposed points (with respect to the w^*-topology). Cf. [10] for the definition of exposed points.

Indeed, if S is a left amenable semigroup and K an invariant subset of βS one may ask the following more general questions: How big is the set $M(S, K)$? Does it have any exposed points? Raimi [12] proved that for each invariant subset K of βN, N the additive semigroup of positive integers, $M(N, K)$ has at least two extreme points. Recently, Fairchild [7], adapting the technique in Granirer [8], proved that if S is a left amenable, countably infinite cancellation semigroup and K is an invariant subset of βS then $M(S, K)$ has infinitely many extreme points. The main result of this paper is the following.

THEOREM. *Let G be a countably infinite amenable group and K be an invariant subset of βG. Then $M(G, K)$ has no exposed points.*

Since every compact convex subset of a finite-dimensional topological vector space has exposed points, the above seemingly negative result implies, among other things, Fairchild's result which we mentioned above.

If X is a discrete set and $\omega \in \beta X$ then ω' will denote the element in $m(X)^*$ defined by $\omega'(f) = f(\omega)$, $f \in m(X)$. If A and B are sets, $A \triangle B = (A \backslash B) \cup (B \backslash A)$ and $|A|$ = number of elements in A.

2. Throughout this section G denotes a countably infinite amenable group with a fixed sequence of finite subsets, F_n, such that

(F1) $F_n \subset F_{n+1}$, $n = 1, 2, \cdots$; $\bigcup F_n = G$,

(F2) $F_n = F_n^{-1}$, $n = 1, 2, \cdots$,

(F3) $\lim_n |xF_n \triangle F_n| / |F_n| = 0$, $x \in X$.

The existence of such a sequence F_n for every countable amenable group is proved by Namioka [11].

For a positive integer n, $T_n : m(G) \to m(G)$ is defined by

$$(T_n f)(\omega) = (1/|F_n|) \sum_{x \in F_n} f(x^{\sim} w), \qquad f \in m(G),\ w \in \beta G.$$

Note that $(T_n f)(\omega) = (T_n^* \omega')f$. For $w \in \beta G$, set Q_ω = the set of w^*-cluster points of the sequence $T_n^* \omega'$.

LEMMA 1. *Let K be an invariant subset of βG and $\omega \in K$. Then $Q_\omega \subset M(G, K)$.*

PROOF. Let $\phi=\lim_\alpha T^*_{n_\alpha}\omega'\in Q_\omega$. Then, for $f\in m(G)$ and $x\in G$,

$$\begin{aligned}
&| (T^*_{n_\alpha}\omega')f - (T^*_{n_\alpha}\omega')l_xf | \\
&\quad = (1/|F_{n_\alpha}|)\left| \sum_{y\in F_{n_\alpha}} f(y^\frown\omega) - \sum_{y\in F_{n_\alpha}} f(x^\frown y^\frown\omega)\right| \\
&\quad \leqq (|F_{n_\alpha}\triangle xF_{n_\alpha}|/|F_{n_\alpha}|)\cdot\|f\| \to 0 \quad \text{as } n_\alpha\to\infty, \quad \text{by (F3)}.
\end{aligned}$$

Thus $\phi(l_xf)=\phi(f)$, i.e., $\phi\in ML(G)$. The fact that suppt $M_\phi\subset K$ is obviously true.

LEMMA 2. *Let ω_n be a sequence of distinct elements in βX where X is an infinite discrete set. Let $n_1<n_2<\cdots$ be an increasing sequence of positive integers. Then the sequence*

$$(1/n_j)(\omega_1'+\omega_2'+\cdots+\omega_{n_j}'), \qquad j=1,2,\cdots,$$

is not convergent in the w^*-topology and hence has at least two w^*-cluster points.

PROOF. When $n_j=j$ this lemma is the main theorem of Rudin [**13**]. His proof also works for this slightly generalized proposition.

The following mean ergodic theorem is similar to that of Calderón [**1**] and Tempel'man [**14**]. If B is a Banach space, $\mathcal{L}(B)$ will denote the algebra of bounded linear operators from B into itself.

LEMMA 3. *Let B be a Banach space and $\mathfrak{U}$ be a mapping of G into $\mathcal{L}(B)$ such that* (1) $\mathfrak{U}^{xy}=\mathfrak{U}^y\mathfrak{U}^x$, $x, y\in G$, (2) $\|\mathfrak{U}^x\|\leqq C$, *a constant, for all $x\in G$. Assume $A\subset B$ is weakly compact and $\mathfrak{U}^xA\subset A$ for $x\in G$. Then for each $f\in A$ the sequence $P_n(f)=(1/|F_n|)\sum_{x\in F_n}\mathfrak{U}^xf$ converges in norm.*

PROOF. The proof is similar to that of [**1**]. We only give a sketch here. Choose a sequence n_j such that $P_{n_j}(f)$ converges to an element $f_0\in A$ is weak topology. Note that we have the following two inequalities: For $x\in G$,

$$\begin{aligned}
\|\mathfrak{U}^x(P_nf) - P_nf\| &= (1/|F_n|)\left\| \sum_{y\in F_n}\mathfrak{U}^{yx}f - \sum_{y\in F_n}\mathfrak{U}^yf\right\| \\
&\leqq (C|F_nx\triangle F_n|/|F_n|)\cdot\|f\| \\
&= (C|x^{-1}F_n\triangle F_n|/|F_n|)\cdot\|f\| \qquad \text{(by (F2))}, \\
\|P_n(\mathfrak{U}^xf) - P_nf\| &\leqq (C|xF_n\triangle F_n|/|F_n|)\cdot\|f\|.
\end{aligned}$$

By (F3) and the above two inequalities, $\lim_n\|\mathfrak{U}^x(P_nf)-P_nf\|=0$ and

$\lim_n \|P_n(\mathcal{U}^x f) - P_n f\| = 0$ $(x \in G)$. Using these two equalities, it is not difficult to conclude that $\lim P_n f = f_0$ in norm.

Clearly the above lemma also holds if we change (1) into (1)′: $\mathcal{U}^{xy} = \mathcal{U}^x \mathcal{U}^y$, $x, y \in G$.

The following lemma is a refinement of [9, Theorem 4].

LEMMA 4. *Let K be an invariant subset of βG. Then for each $f \in m(G)$ there exist $\omega \in K$ and $\phi_i \in Q_\omega$, $i = 1, 2$, $\phi_1 \neq \phi_2$, such that*

$$\sup\{\phi(f) : \phi \in M(G, K)\} = \phi_1(f) = \phi_2(f).$$

PROOF. Denote $\sup\{\phi(f) : \phi \in M(G, K)\}$ by $\alpha(f)$. Since $M(G, K)$ is nonempty and w^*-compact there exists $\phi \in M(G, K)$ such that $\phi(f) = \alpha(f)$. Let $B = L^2(\mu_\phi)$ and $\mathcal{U}^x$ be defined by $(\mathcal{U}^x h)(\omega) = h(x^\sim \omega)$, $h \in L^2(\mu_\phi)$, $\omega \in K$. Since ϕ is left invariant, $\|\mathcal{U}^x\| = 1$, $x \in G$, and clearly $\mathcal{U}^{xy} = \mathcal{U}^y \mathcal{U}^x$. Also note that, if $f \in m(G)$,

$$(P_n f)(\omega) = (1/|F_n|) \sum_{y \in F_n} f(y^\sim \omega) = (T_n f)(\omega), \qquad \omega \in K.$$

Therefore, by Lemma 3, $\lim_n T_n f$ exists in $L^2(\mu_\phi)$ norm. Denote the limit by f_0. Choose a subsequence n_j such that $\lim_j (T_{n_j} f)(\omega) = f_0(\omega)$ exists for $\omega \in D \subset K$ where D is a Borel subset of K and $\mu_\phi(D) = 1$. Thus

$$\text{(i)} \qquad \int f_0 \, d\mu_\phi = \lim_j \int (T_{n_j} f) \, d\mu_\phi = \phi(T_{n_j} f) = \phi(f).$$

Also note that if $\omega \in D$ then there exists $\psi \in Q_\omega$ such that $\psi(f) = f_0(\omega)$. Hence by Lemma 1

$$\text{(ii)} \qquad f_0(\omega) \leqq \alpha(f) = \phi(f).$$

Compare (i) and (ii) we see that $f_0(\omega) = \phi(f)$ for almost all $\omega \in D$. In particular, there exists $\omega_0 \in K$ such that

$$\text{(iii)} \qquad \phi(f) = \lim_j (T_{n_j} f)(\omega_0).$$

Finally, note that if $x, y \in G$, $x \neq y$, then $x^\sim \omega_0 \neq y^\sim \omega_0$ [3, Lemma 1]. Thus we may apply Lemma 2 to the set $\{x^\sim \omega_0 : x \in G\}$ and conclude that the sequence

$$(1/|F_{n_j}|) \sum_{x \in F_{n_j}} (x^\sim \omega_0)' = T^*_{n_j} \omega'$$

has at least two ω^*-cluster points ϕ_1 and ϕ_2. By the definition, $\phi_i \in Q_{\omega_0}$, $i = 1, 2$. That $\phi_1(f) = \phi_2(f) = \alpha(f)$ follows directly from (iii).

THEOREM 1. *Let G be a countably infinite amenable group and let K be an invariant subset of βG. Then the w^*-compact convex set $M(G, K)$ has no exposed points.*

PROOF. $\phi \in M(G, K)$ is an exposed point if and only if there exists $f \in M(G)$ such that $\phi(f) > \psi(f)$ for $\psi \in M(G, K)$, $\psi \neq \phi$. By Lemma 4, there is no $\phi \in M(G, K)$ which has this property.

When $K = \beta G$, $M(G, K) = ML(G)$. We want to state this special case separately.

COROLLARY 1. *Let G be a countably infinite amenable group. Then $ML(G)$ has exactly 2^c extreme points but has no exposed points.*

That $ML(G)$ has 2^c extreme points is contained in [**2**].

COROLLARY 2. *Let S be a left amenable countably infinite cancellation semigroup.*

(1) *If K is a minimal invariant subset of βS then $M(S, K)$ has no exposed points.*

(2) *If K is an invariant subset of βS then $M(S, K)$ cannot be embedded into a Banach space affinely and topologically. In particular, $M(S, K)$ has to be infinite dimensional.*

PROOF. (1) Since S is cancellative it can be considered as a subsemigroup of an amenable group G (cf. [**15**]). We may assume that G is generated by S. In particular, G is also countably infinite. Let K be a minimal invariant subset of βS. Fix any $\phi \in M(S, K)$. For $s \in S$, using the fact that $s^{\sim}$ is one-one [**2**, Lemma 2.1], one gets that $\mu_\phi(s^{\sim}K) = \mu_\phi(K) = 1$. Therefore suppt $\mu_\phi \subset s^{\sim}K \subset K$. On the other hand, since suppt μ_ϕ is invariant [**15**, Theorem 4.3] and K is minimal invariant we conclude that $K = s^{\sim}K =$ suppt μ_ϕ. Since G is generated by S we see that $x^{\sim}K = K$ for each $x \in G$, i.e., K is an invariant subset of βG. Therefore, by Theorem 1, $M(G, K)$ has no exposed points. It is easily checked that $M(G, K) = M(S, K)$. Thus $M(S, K)$ has no exposed points.

(2) Let K be an invariant subset of βS. By Zorn's Lemma, K contains a minimal invariant subset K_1. Hence, by (1), $M(S, K_1)$ has no exposed points. Note that $M(S, K_1) \subset M(S, K)$. Thus the result follows from the well-known fact that every compact convex subset of a Banach space has exposed points (cf. Klee [**10**]).

Let G be an amenable group and H a homomorphic image of G. Then it is known that if, for each invariant subset K_1 of βH, $M(H, K_1)$ is infinite dimensional then $M(G, K)$ is also infinite dimensional for each invariant subset K of βG [**7**, Proposition 5.7]. In particular, if G is an infinite abelian group then G has a countably infinite homomorphic image H [**13**]. Thus by Corollary 2 we have the following.

COROLLARY 3. *Let S be an infinite abelian cancellation semigroup and K an invariant subset of βS. Then $M(S, K)$ is infinite dimensional.*

Another consequence of Lemma 4 is the following generalization of Theorem 4 in [**9**].

THEOREM 2. *Let G be a countable amenable group with a sequence of finite sets F_n which satisfies* (F1), (F2) *and* (F3). *Then $ML(G)$ equals the ω^*-closed convex hull of* $\bigcup \{Q_\omega : \omega \in \beta G\}$.

3. **Remarks.** (1) We believe that Theorem 1 holds for every infinite amenable group. But we do not know how to prove it.

When G is a countably infinite amenable group and K an invariant subset of βG then Theorem 1 tells us that $M(G, K)$ has infinitely many extreme points. It is interesting to know exactly how many extreme points are in $M(G, K)$. Are there 2^c of them?

(2) Let G be a unimodular σ-compact locally compact amenable group. Then, same as the discrete case, there exists a sequence F_n of compact neighborhoods of the identity such that (F1), (F2) and (F3) hold [**4**, Theorem 4]. Of course, here in (F3), $|A|$ denotes the Haar measure of a set A, instead of the number of elements in A. The mean ergodic theorem (Lemma 3) also holds for the above G and F_n: Let $x \to \mathfrak{U}^x$ be a weakly continuous homomorphism of G into $\mathfrak{L}(B)$, B a Banach space, such that $\|\mathfrak{U}^x\| \leqq C$ for each $x \in G$ where C is a fixed constant. Suppose there is a weakly compact convex set $A \subset B$ such that $\mathfrak{U}^x A \subset A$, $x \in G$. Then for each $u \in A$, $(1/|F_n|)\int_{F_n} \mathfrak{U}^x(u)dx$ converges in norm to an element in A (cf. Calderón [**1**] and Tempel'man [**14**]).

[**1**] and [**14**] also contain an individual ergodic theorem with respect to a sequence similar to F_n above with an additional condition: there exists $k > 0$ such that

$$\text{(E)} \qquad |F_n^2| \leqq k|F_n|, \qquad n = 1, 2, \cdots.$$

We do not know whether the individual ergodic theorem holds without (E). Even for a countable amenable discrete group it is unlikely in general that a sequence F_n can be found to satisfy (F1), (F2), (F3) and (E) simultaneously.

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