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SHORTER NOTES

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NOTE ON SALIÉ'S SUM

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ABSTRACT. It is shown in a very simple way that an exponential sum (involving the Legendre symbol) considered by Salié is the sum of two Gauss sums.

Let p denote an odd prime. Whenever we write $\sum_x$ the summation is taken over all x in a complete residue system modulo p. If we write $\sum_x'$ the summation is over all x in a reduced residue system modulo p. For x in a reduced residue system $\bar{x}$ denotes its inverse modulo p.

For integers a and b such that $ab \not\equiv 0$ (all congruences are modulo p), Salié's sum $S_p(a, b)$ is defined by

$$S_p(a, b) = \sum_x{}' \left(\frac{x}{p}\right) \exp(2\pi i(ax + b\bar{x})/p), \tag{1}$$

where (x/p) is Legendre's symbol of quadratic residuacity modulo p. If $(ab/p) = -1$ applying the mapping $x \to \bar{a}b\bar{x}$ to Salié's sum (1) gives $S_p(a, b) = -S_p(a, b)$, so that $S_p(a, b) = 0$. If $(ab/p) = +1$, say $ab \equiv c^2 \pmod{p}$, applying the mapping $x \to \bar{a}cx$ gives $S_p(a, b) = (ac/p)S_p(c, c)$. In 1931 Salié [3] showed that $S_p(c, c)$ can be evaluated explicitly. He proved that

$$S_p(c, c) = 2\left(\frac{c}{p}\right) i^{((p-1)/2)^2} p^{1/2} \cos(4\pi c/p).$$

The author [4], [5] was the first to explain why $S_p(c, c)$ can be evaluated explicitly by showing that it is the sum of two Gaussian sums. (Other evaluations have been given by Mordell [1], [2].) The following is perhaps the simplest known proof of this result.

For $y \not\equiv 2$ we have

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$$\left(\frac{y-2}{p}\right) \sum_{x;\, x+\bar{x}\equiv y}' \left(\frac{x}{p}\right) = \sum_{x;\, x+\bar{x}\equiv y}' \left(\frac{x(x+\bar{x}-2)}{p}\right)$$
$$= \sum_{x;\, x^2-yx+1\equiv 0} \left(\frac{(x-1)^2}{p}\right)$$
$$= \sum_{x;\, x^2-yx+1\equiv 0} 1 = 1 + \left(\frac{y^2-4}{p}\right),$$

so that

$$\sum_{x;\, x+\bar{x}\equiv y}' \left(\frac{x}{p}\right) = \left(\frac{y-2}{p}\right) + \left(\frac{y+2}{p}\right). \tag{2}$$

Clearly (2) is also true if $y \equiv 2 \pmod{p}$, and so we have

$$S_p(c, c) = \sum_y \left\{ \sum_{x;\, x+\bar{x}\equiv y}' \left(\frac{x}{p}\right) \right\} e(cy)$$
$$= \sum_y \left(\frac{y-2}{p}\right) e(cy) + \sum_y \left(\frac{y+2}{p}\right) e(cy).$$

This gives $S_p(c, c)$ as the sum of the two Gauss sums

$$\sum_y \left(\frac{y \pm 2}{p}\right) e(cy) = \left(\frac{c}{p}\right) i^{((p-1)/2)^2} p^{1/2} e(\mp 2c),$$

as required.

References

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4. K. S. Williams, *Finite transformation formulae involving the Legendre symbol*, Pacific J. Math. **34** (1970), 559–568.

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