A DIFFERENTIATION THEOREM FOR FUNCTIONS DEFINED ON THE DYADIC RATIONALS

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Abstract. In this paper we show that under certain conditions a real-valued function defined on an interval of dyadic rational numbers is a monotone function. One of these conditions involves a generalized differentiability property. From this result we offer a new proof of a conjecture of N. Fine concerning the uniqueness of solution of Walsh series.

1. Introduction. Let \( f \) be a real-valued function defined on \((a, b) \cap D, a < b\), where \( D \) denotes the set of dyadic rational numbers. In this paper we shall describe conditions on \( f \) which will ensure that \( f \) is monotone decreasing on \((a, b) \cap D\). This will enable us to reprove a conjecture of N. Fine [2] concerning the uniqueness of solution of Walsh series.

For our purposes we introduce the following functions.

Given any nonnegative integer \( n \) and real number \( x \), let \( \alpha_n(x) = k/2^n, \beta_n(x) = (k+1)/2^n \) where \( k \) is that integer for which \( k \leq 2^n x < k + 1 \). Also set

\[
\alpha'_n(x) = \alpha_n(x), \quad x \in D,
\]

\[
= \alpha_n(x) - 1/2^n, \quad x \not\in D.
\]

Now given any \( x \) in \((a, b)\) we write

\[
D_n f(x) = \limsup_{n \to \infty} \left[ f(\beta_n(x)) - f(\alpha_n(x)) \right] \cdot 2^n.
\]

2. The main theorem. Let \( G \) be a real-valued function defined on \((a, b) \cap D\). Our primary result can be stated as follows.

Theorem. Assume that \( G \) satisfies the following conditions:

(i) \( \limsup_{n \to \infty} G(\alpha'_n(x)) = G(x), \quad x \in (a, b) \cap D. \)

(ii) \( \liminf_{n \to \infty} [G(\beta_n(x)) - G(\alpha_n(x))] \leq 0, \quad x \in (a, b). \)

(iii) \( D_n G(x) \leq 0, \quad x \in (a, b) \setminus E \) for some countable set \( E \).

Then \( G \) is monotone decreasing on \((a, b) \cap D\).

Proof. Clearly one may assume that \( E = \{ x_k \}_{k=1}^\infty \) contains the

Received by the editors June 10, 1969 and, in revised form, November 9, 1970.
AMS 1969 subject classifications. Primary 2640, 2650; Secondary 4211.
Key words and phrases. Monotone functions on dyadic rationals, Walsh series, Walsh-Fourier series.

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dyadic rationals in \((a, b)\). Moreover, by considering the functions
\[ G_\epsilon(x) = G(x) - \epsilon x, \ x \in (a, b) \]
for each \(\epsilon > 0\) we see that one may assume \(G\) satisfies
\[ (iv) \quad \liminf_{n \to \infty} 2^n \left[ G(\beta_n(x)) - G(\alpha_n(x)) \right] < -\epsilon \]
for some \(\epsilon > 0\) and all \(x\) in \((a, b)\)\(\setminus\)\(E\). Now once we have established that
\[ G(k/2^p) \leq G((k - 1)/2^p), \quad (k - 1)/2^p, k/2^p \in (a, b), \]
where \(k\) and \(p\) are integers, \(p \geq 0\), the theorem follows. Hence assume to the contrary that there exists integers \(k_0\) and \(p\), \(p \geq 0\), with \((k_0 - 1)/2^p, k_0/2^p \in (a, b)\) such that
\[ G(k_0/2^p) > G((k_0 - 1)/2^p). \]
Set \(\xi_0 = (k_0 - 1)/2^p, \eta_0 = k_0/2^p, I_0 = [\xi_0, \eta_0]\). It will prove convenient to define a set function \(\mu\) by:
\[ \mu([\xi, \eta]) = G(\eta) - G(\xi), \quad \xi, \eta \in D \cap (a, b). \]
Then \(\mu(I_0) > 0\) by assumption. Let \(E_0 = E\) and define \(I_n, E_n\) inductively by the following procedure. Having chosen \(I_n = [\xi_n, \eta_n]\), \(\xi_n, \eta_n \in (a, b) \cap D\) with \(\mu(I_n) > 0\) and \(E_n = E \cap I_n\), let
\[ I_n^1 = \left[ \xi_n, \frac{\xi_n + \eta_n}{2} \right], \quad I_n^2 = \left[ \frac{\xi_n + \eta_n}{2}, \eta_n \right]. \]
If \(E_n\) is empty, set \(I_{n+1} = I_n^i\) where \(i \in \{1, 2\}\) is smallest possible such that \(\mu(I_n^i) > 0\). If \(E_n\) is nonempty, let \(n'\) be the smallest subscript such that \(\mu(I_n^i) > 0\). Then assume first that \(x_n'\) is the midpoint of \(I_n^i\). Then set \(I_{n+1} = I_n^i\) where \(i \in \{1, 2\}\) is smallest possible such that \(\mu(I_n^i) > 0\). If \(x_n'\) is not the midpoint of \(I_n\), one has \(x_n' \in I_n^i, x_n' \in I_n^j, \{i, j\} = \{1, 2\}\). Set \(I_{n+1} = I_n^i\) if \(\mu(I_n^i) > 0\). Otherwise set \(I_{n+1} = I_n^j\). This defines \(I_{n+1} = [\xi_{n+1}, \eta_{n+1}]\), \(\xi_{n+1}, \eta_{n+1} \in (a, b) \cap D\). Set \(E_{n+1} = E \cap I_{n+1}\). In each of the above cases we have \(\mu(I_{n+1}) > 0\) since \(\mu(I_n) = \mu(I_n^i) + \mu(I_n^j) > 0\).

Observe that for each \(n, I_n\) has length \(2^{-(p+n)}\). Moreover if \(I_{n+1} = I_n^i\) and \(\mu(I_n^i) \leq 0\), then \(\mu(I_{n+1}) \leq \mu(I_n)\) where \(\{i, j\} = \{1, 2\}\). Set \(\{x\} = \bigcap_{n=0}^{\infty} I_n\) and notice that \(\bigcap_{n=0}^{\infty} E_n = E \cap \{x\}\).

Case 1. \(x \in E \cap D\). Then there exists \(N > 0\) such that for each \(n > N, x\) is the left (or right) endpoint of \(I_n\). Suppose that \(x\) is the left endpoint for \(n > N\). Then \(I_n = [\alpha_{n+p}(x), \beta_{n+p}(x)], I_n = I_{n-1}^1, \mu(I_{n-1}^1) \leq 0\). Hence \(\mu(I_{n-1}) \leq \mu(I_n)\) for all \(n > N\) which implies that
\[ \liminf_{n \to \infty} [G(\beta_n(x)) - G(\alpha_n(x))] > 0. \]
This contradicts (ii). Assume next that $x$ is a right endpoint for $n > N$. Again one has $\mu(I_{n-1}) \leq \mu(I_n)$ and $I_n = [\alpha'_{n+p}(x), \beta_{n+p}(x)]$ so

$$\lim \inf_{n \to \infty} (G(x) - G(\alpha_n(x))) > 0$$

which contradicts (i).

Case 2. $x \in E$, $x \notin D$. Then for each $n$, $I_n = [\alpha_{n+p}(x), \beta_{n+p}(x)]$ and $\mu(I_n) \leq \mu(I_{n+1})$ which again contradicts (ii).

Case 3. $x \notin E$. As in Case 2 we can write $I_n = [\alpha_{n+p}(x), \beta_{n+p}(x)]$ for all $n$. Consequently,

$$\lim \inf_{n \to \infty} 2^n(G(\beta_n(x)) - G(\alpha_n(x))) \geq 0$$

which contradicts (iv). This completes the proof.

It should be remarked that this result improves the lemma of N. Fine as given in [2, p. 407].

3. An application to Walsh series. In 1947, N. Fine in his classical paper on Walsh-Fourier series [2] considered the problem of determining when a Walsh series is the Walsh-Fourier series of a Lebesgue integrable function. It was conjectured that given a Walsh series which converges to an integrable function except on a countable set, the series is the Walsh-Fourier series of the function. This conjecture was proved in 1964 by R. Crittenden and V. Shapiro in [1]. Their proof was rather lengthy and involved an intricate application of the Baire category theorem. We offer a simplified proof using only the results in [2] together with the main theorem.

We now introduce some standard terminology and restate certain theorems from [2] which will be used in the sequel. Let $\psi_k$ denote the $k$th Walsh function on the interval $[0, 1]$ and set $J_k(x) = \int_0^x \psi_k(t) \, dt$, $x \in [0, 1]$, for $k = 0, 1, \cdots$.

**Theorem 1.** If $\sum_{k=0}^{\infty} a_k \psi_k(x)$ converges at $x$, then so does $\sum_{k=0}^{\infty} a_k J_k(x)$.

From [2, p. 405] one has with slight modifications:

**Theorem 2.** If $(a_k)_{k=0}^{\infty}$ converges to zero and $L(x) = \sum_{k=0}^{\infty} a_k J_k(x)$ defines an essentially absolutely continuous function on $[0, 1]$, then $\sum_{k=0}^{\infty} a_k \psi_k(x)$ is the Walsh-Fourier series of $L'(x)$.

From the two theorems in [2, p. 406] one has:

**Theorem 3.** Let $(a_k)_{k=0}^{\infty}$ converge to zero, set $L(x) = \sum_{k=0}^{\infty} a_k J_k(x)$, $x \in [0, 1]$. Then this series converges for each $x \in D \cap [0, 1]$. Moreover for
each \( x \in [0, 1] \), \( L(\beta_n(x)) - L(\alpha_n(x)) = 2^{-n}S_2(x) \) which converges to zero uniformly in \( x \).

From these results together with the main theorem we now prove the conjecture.

**Theorem 4.** Let \( \sum_{n=0}^{\infty} a_n \psi_n(x) \) converge to a finite-valued integrable function \( f \) except on a countable set of points \( E \) in \([0, 1]\). Then this series is the Walsh-Fourier series of \( f \).

**Proof.** Set \( F(x) = \int_{0}^{x} f(t) \, dt \), \( x \in [0, 1] \), and fix \( \epsilon > 0 \). By the Vitali-Caratheodory theorem \([3, \text{p. 75}]\) one can select two absolutely continuous functions \( \phi_\epsilon \) and \( \psi_\epsilon \) on \([0, 1]\) such that

\[
| \phi_\epsilon(x) - F(x) | < \epsilon, \quad | \psi_\epsilon(x) - F(x) | < \epsilon, \quad x \in [0, 1],
\]

and the derivatives of \( \phi_\epsilon(x) \) (resp. \( \psi_\epsilon(x) \)) are less than (resp. greater than) \( f(x) \) whenever \( f(x) \neq -\infty \) (resp. \( f(x) \neq +\infty \)). For \( x \) in \( D \cap (0, 1) \), set \( G_\epsilon(x) = \phi_\epsilon(x) - L(x) \), \( H_\epsilon(x) = L(x) - \psi_\epsilon(x) \). The functions \( G_\epsilon \) and \( H_\epsilon \) exist by Theorem 3. By Theorem 3, \( G_\epsilon \) and \( H_\epsilon \) satisfy the hypothesis in the lemma. Hence \( G_\epsilon \) and \( H_\epsilon \) are monotone decreasing on \( D \cap (0, 1) \). Letting \( \epsilon \) tend to zero one has that \( F - L \) and \( L - F \) are monotone decreasing on \( (0, 1) \cap D \). Hence \( F - L \) is constant on \( (0, 1) \cap D \) so setting \( L(x) = F(x) + c, \ x \in (0, 1) \cap D \), we see that this equality extends by Theorem 3 to all \( x \) for which \( L(x) \) exists. By Theorem 1, \( L(x) \) exists for essentially all \( x \in E \). Consequently, \( L \) is essentially absolutely continuous on \([0, 1]\) so, by Theorem 2, \( \sum_{n=0}^{\infty} a_n \psi_n(x) \) is the Walsh-Fourier series of \( L'(x) = f(x) \).

**Bibliography**


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