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A DIFFERENTIATION THEOREM FOR FUNCTIONS DEFINED ON THE DYADIC RATIONALS

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ABSTRACT. In this paper we show that under certain conditions a real-valued function defined on an interval of dyadic rational numbers is a monotone function. One of these conditions involves a generalized differentiability property. From this result we offer a new proof of a conjecture of N. Fine concerning the uniqueness of solution of Walsh series.

1. **Introduction.** Let f be a real-valued function defined on $(a, b) \cap D$, $a<b$, where D denotes the set of dyadic rational numbers. In this paper we shall describe conditions on f which will ensure that f is monotone decreasing on $(a, b)\cap D$. This will enable us to reprove a conjecture of N. Fine [2] concerning the uniqueness of solution of Walsh series.

For our purposes we introduce the following functions.

Given any nonnegative integer n and real number x, let $\alpha_n(x) = k/2^n$, $\beta_n(x) = (k+1)/2^n$ where k is that integer for which $k \leqq 2^n x < k+1$. Also set

$$\begin{aligned} \alpha'_n(x) &= \alpha_n(x), & x &\notin D, \\ &= \alpha_n(x) - 1/2^n, & x &\in D. \end{aligned}$$

Now given any x in (a, b) we write

$$D\_f(x) = \liminf_{n\to\infty} [f(\beta_n(x)) - f(\alpha_n(x))] \cdot 2^n.$$

2. **The main theorem.** Let G be a real-valued function defined on $(a, b)\cap D$. Our primary result can be stated as follows.

THEOREM. *Assume that G satisfies the following conditions:*
(i) $\limsup_{n\to\infty} G(\alpha'_n(x)) \geqq G(x)$, $x \in (a, b)\cap D$.
(ii) $\liminf_{n\to\infty} [G(\beta_n(x)) - G(\alpha_n(x))] \leqq 0$, $x \in (a, b)$.
(iii) $D\_G(x) \leqq 0$, $x \in (a, b)\setminus E$ *for some countable set E.*
Then G is monotone decreasing on $(a, b)\cap D$.

PROOF. Clearly one may assume that $E = \{x_k\}_{k=1}^{\infty}$ contains the

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dyadic rationals in (a, b). Moreover, by considering the functions $G_\epsilon(x) = G(x) - \epsilon x$, $x \in (a, b)$, for each $\epsilon > 0$ we see that one may assume G satisfies

(iv) $$\liminf_{n\to\infty} 2^n[G(\beta_n(x)) - G(\alpha_n(x))] < -\epsilon$$

for some $\epsilon > 0$ and all x in $(a, b)\backslash E$. Now once we have established that

$$G(k/2^p) \leqq G((k-1)/2^p), \qquad (k-1)/2^p, k/2^p \in (a, b),$$

where k and p are integers, $p \geqq 0$, the theorem follows. Hence assume to the contrary that there exists integers k_0 and p, $p \geqq 0$, with $(k_0 - 1)/2^p, k_0/2^p \in (a, b)$ such that

$$G(k_0/2^p) > G((k_0 - 1)/2^p).$$

Set $\xi_0 = (k_0 - 1)/2^p$, $\eta_0 = k_0/2^p$, $I_0 = [\xi_0, \eta_0]$. It will prove convenient to define a set function μ by:

$$\mu([\xi, \eta]) = G(\eta) - G(\xi), \qquad \xi, \eta \in D \cap (a, b).$$

Then $\mu(I_0) > 0$ by assumption. Let $E_0 = E$ and define I_n, E_n inductively by the following procedure. Having chosen $I_n = [\xi_n, \eta_n]$, $\xi_n, \eta_n \in (a, b) \cap D$ with $\mu(I_n) > 0$ and $E_n = E \cap I_n$, let

$$I_n^1 = \left[\xi_n, \frac{\xi_n + \eta_n}{2}\right], \qquad I_n^2 = \left[\frac{\xi_n + \eta_n}{2}, \eta_n\right].$$

If E_n is empty, set $I_{n+1} = I_n^i$ where $i \in \{1, 2\}$ is smallest possible such that $\mu(I_n^i) > 0$. If E_n is nonempty, let n' be the smallest subscript such that $x_{n'}$ is in E_n. Assume first that $x_{n'}$ is the midpoint of I_n. Then set $I_{n+1} = I_n^i$ where $i \in \{1, 2\}$ is smallest possible such that $\mu(I_n^i) > 0$. If $x_{n'}$ is not the midpoint of I_n, one has $x_{n'} \in I_n^i$, $x_{n'} \notin I_n^j$, $\{i, j\} = \{1, 2\}$. Set $I_{n+1} = I_n^j$ if $\mu(I_n^j) > 0$. Otherwise set $I_{n+1} = I_n^i$. This defines $I_{n+1} = [\xi_{n+1}, \eta_{n+1}]$, $\xi_{n+1}, \eta_{n+1} \in (a, b) \cap D$. Set $E_{n+1} = E \cap I_{n+1}$. In each of the above cases we have $\mu(I_{n+1}) > 0$ since $\mu(I_n) = \mu(I_n^1) + \mu(I_n^2) > 0$.

Observe that for each n, I_n has length $2^{-(p+n)}$. Moreover if $I_{n+1} = I_n^i$ and $\mu(I_n^j) \leqq 0$, then $\mu(I_{n+1}) \geqq \mu(I_n)$ where $\{i, j\} = \{1, 2\}$. Set $\{x\} = \bigcap_{n=0}^\infty I_n$ and notice that $\bigcap_{n=1}^\infty E_n = E \cap \{x\}$.

Case 1. $x \in E \cap D$. Then there exists $N > 0$ such that for each $n > N$, x is the left (or right) endpoint of I_n. Suppose that x is the left endpoint for $n > N$. Then $I_n = [\alpha_{n+p}(x), \beta_{n+p}(x)]$, $I_n = I_{n-1}^1$, $\mu(I_{n-1}^2) \leqq 0$. Hence $\mu(I_{n-1}) \leqq \mu(I_n)$ for all $n > N$ which implies that

$$\liminf_{n\to\infty} [G(\beta_n(x)) - G(\alpha_n(x))] > 0.$$

This contradicts (ii). Assume next that x is a right endpoint for $n>N$. Again one has $\mu(I_{n-1})\leqq\mu(I_n)$ and $I_n=[\alpha'_{n+p}(x), x]$ so

$$\liminf_{n\to\infty}[G(x)-G(\alpha'_n(x))]>0$$

which contradicts (i).

Case 2. $x\in E$, $x\notin D$. Then for each n, $I_n=[\alpha_{n+p}(x), \beta_{n+p}(x)]$ and $\mu(I_n)\leqq\mu(I_{n+1})$ which again contradicts (ii).

Case 3. $x\notin E$. As in Case 2 we can write $I_n=[\alpha_{n+p}(x), \beta_{n+p}(x)]$ for all n. Consequently,

$$\liminf_{n\to\infty} 2^n[G(\beta_n(x))-G(\alpha_n(x))]\geqq 0$$

which contradicts (iv). This completes the proof.

It should be remarked that this result improves the lemma of N. Fine as given in [2, p. 407].

3. **An application to Walsh series.** In 1947, N. Fine in his classical paper on Walsh-Fourier series [2] considered the problem of determining when a Walsh series is the Walsh-Fourier series of a Lebesgue integrable function. It was conjectured that given a Walsh series which converges to an integrable function except on a countable set, the series is the Walsh-Fourier series of the function. This conjecture was proved in 1964 by R. Crittenden and V. Shapiro in [1]. Their proof was rather lengthy and involved an intricate application of the Baire category theorem. We offer a simplified proof using only the results in [2] together with the main theorem.

We now introduce some standard terminology and restate certain theorems from [2] which will be used in the sequel. Let ψ_k denote the kth Walsh function on the interval $[0, 1]$ and set $J_k(x)=\int_0^x\psi_k(t)dt$, $x\in[0, 1]$, for $k=0, 1, \cdots$.

THEOREM 1. *If* $\sum_{k=0}^{\infty} a_k\psi_k(x)$ *converges at* x, *then so does* $\sum_{k=0}^{\infty} a_kJ_k(x)$.

From [2, p. 405] one has with slight modifications:

THEOREM 2. *If* $(a_k)_{k=0}^{\infty}$ *converges to zero and* $L(x)\equiv\sum_{k=0}^{\infty} a_kJ_k(x)$ *defines an essentially absolutely continuous function on* $[0, 1]$, *then* $\sum_{k=0}^{\infty} a_k\psi_k(x)$ *is the Walsh-Fourier series of* $L'(x)$.

From the two theorems in [2, p. 406] one has:

THEOREM 3. *Let* $(a_k)_{k=0}^{\infty}$ *converge to zero, set* $L(x)\equiv\sum_{k=0}^{\infty} a_kJ_k(x)$, $x\in[0, 1]$. *Then this series converges for each* $x\in D\cap[0, 1]$. *Moreover for* *each* $x \in [0, 1]$, $L(\beta_n(x)) - L(\alpha_n(x)) = 2^{-n}S_{2^n}(x)$ *which converges to zero uniformly in* x.

From these results together with the main theorem we now prove the conjecture.

THEOREM 4. *Let* $\sum_{k=0}^{\infty} a_k\psi_k(x)$ *converge to a finite-valued integrable function f except on a countable set of points E in* $[0, 1]$. *Then this series is the Walsh-Fourier series of f.*

PROOF. Set $F(x) = \int_0^x f(t)dt$, $x \in [0, 1]$, and fix $\epsilon > 0$. By the Vitali-Carathéodory theorem [**3**, p. 75] one can select two absolutely continuous functions ϕ_ϵ and ψ_ϵ on $[0, 1]$ such that

$$|\phi_\epsilon(x) - F(x)| < \epsilon, \quad |\psi_\epsilon(x) - F(x)| < \epsilon, \qquad x \in [0, 1],$$

and the derivates of $\phi_\epsilon(x)$ (resp. $\psi_\epsilon(x)$) are less than (resp. greater than) $f(x)$ whenever $f(x) \neq -\infty$ (resp. $f(x) \neq +\infty$). For x in $D \cap (0, 1)$, set $G_\epsilon(x) = \phi_\epsilon(x) - L(x)$, $H_\epsilon(x) = L(x) - \psi_\epsilon(x)$. The functions G_ϵ and H_ϵ exist by Theorem 3. By Theorem 3, G_ϵ and H_ϵ satisfy the hypothesis in the lemma. Hence G_ϵ and H_ϵ are monotone decreasing on $D \cap (0, 1)$. Letting ϵ tend to zero one has that $F - L$ and $L - F$ are monotone decreasing on $(0, 1) \cap D$. Hence $F - L$ is constant on $(0, 1) \cap D$ so setting $L(x) = F(x) + c$, $x \in (0, 1) \cap D$, we see that this equality extends by Theorem 3 to all x for which $L(x)$ exists. By Theorem 1, $L(x)$ exists for all $x \notin E$. Consequently, L is essentially absolutely continuous on $[0, 1]$ so, by Theorem 2, $\sum_{k=0}^{\infty} a_k\psi_k(x)$ is the Walsh-Fourier series of $L'(x) = f(x)$.

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