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QUASI-DIFFERENTIABLE FUNCTIONS ON BANACH SPACES

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ABSTRACT. Nonzero Fréchet differentiable functions with bounded support do not exist on certain real separable Banach spaces. As a result, the class of differentiable functions on such spaces is too small to be useful. For example, the class of differentiable functions on certain spaces does not separate disjoint closed subsets of the space. It is shown that this separation problem does not arise if Fréchet differentiability is replaced by the weaker condition of quasi-differentiability. Furthermore, it is shown that any bounded uniformly continuous function on a real separable Banach space is the uniform limit of quasi-differentiable functions.

1. **Introduction.** Let B be a Banach space with norm $\|\cdot\|$ and topological dual space B^*. A real-valued function f on B is said to be *differentiable* at x if there exists an element λ in B^* such that the inequality $|f(x+y)-f(x)-\langle\lambda, y\rangle| \leqq o(\|y\|)$ holds for y in a neighborhood of the origin in B. A function is said to be differentiable on B if the function is differentiable at each point in B. The works [1] and [5] show that for certain Banach spaces the behavior of a differentiable function is quite restricted. In fact, Whitfield [8] has shown that if B is a separable Banach space and B^* is not a separable space in the dual norm then there are no nonzero differentiable functions with bounded support on B. An example of such a space is the Banach space $C[0, 1]$ consisting of all real-valued continuous functions on $[0, 1]$. One may verify directly that $C[0, 1]^*$ contains an isomorphic image of $L^\infty(0, 1)$. Hence, $C[0, 1]^*$ is not separable.

A function f on a space B is said to be *quasi-differentiable* at x if there is an element λ in B^* with the following property. For each continuous map $\alpha:(-1, 1)\rightarrow B$ such that $\alpha(0)=x$ and α is differentiable at zero, the function $f(\alpha(t))$ has a derivative at zero equal to $\langle\lambda, \alpha'(0)\rangle$. This condition is equivalent to a definition given by Dieudonné in [2, p. 151] if f is assumed to be continuous. It is a consequence of the chain rule that any differentiable function is quasi-differentiable.

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Also, it is shown in [**2**, p. 152] that the converse holds if the space is finite dimensional.

We prove that a set of bounded quasi-differentiable functions on any real separable Banach space is a dense subset of the bounded uniformly continuous functions on the space. To establish this result we use certain finite Borel measures which are carried by the Banach space. These measures define smoothing operators which act on bounded continuous functions. Our result is a consequence of the fact, which is proved below, that any function satisfying a Lipschitz condition is smoothed to a quasi-differentiable function by these operators.

We indicate how the above measures arise. Let B be a real separable Banach space with norm $\|\cdot\|$. Gross [**3**] has shown that there exists a continuous linear embedding e of a real separable Hilbert space into B such that the range of e is dense and such that the function $n(x) = \|ex\|$ is a *measurable norm* on the Hilbert space. The concept of a *measurable norm* is defined in [**3**, Definition 4, p. 34] and it is shown there that the space B carries a family of Borel probability measures $\{p_t\}$, $t>0$, which are characterized by the following property: Any element λ of B^* is a Gaussian random variable with mean zero relative to the measure p_t. That is, for each real r there holds

$$p_t(\{x \in B: \langle \lambda, x\rangle < r\}) = (2\pi ta)^{-1/2}\int_{-\infty}^{r} \exp[-(2ta)^{-1}s^2]ds.$$

Also, $a=|e^*\lambda|^2$ where $|\cdot|$ is the norm on the dual of the Hilbert space. The measure p_t is said to be *abstract Wiener measure* on B with variance parameter t. If g is a bounded continuous function on B, we denote by p_tg the function $(p_tg)(x)=\int_B g(x+y)p_t(dy)$. The notation above will be fixed throughout the paper.

2. **Approximation theorem.** Let f be a quasi-differentiable function on a space B. For a fixed x in B, the linear functional which appears in the above definition of quasi-differentiability is unique, and we denote the linear functional by $f'(x)$. This defines a map $f': B \to B^*$ which is said to be the quasi-derivative of f.

DEFINITION. A quasi-differentiable function f on a Banach space B is of *class* Q^1 if f' is bounded in B^* norm and the map $(x, y) \to \langle f'(x), y\rangle$ is continuous on $B \times B$.

THEOREM. *Let B be a real separable Banach space. The set of bounded functions on B of class Q^1 is dense in the space of bounded uniformly continuous functions on B.*

PROOF. Let B be a real separable Banach space with norm $\|\cdot\|$. A function g on B is said to be a Lipschitz function if there exists a constant $C>0$ such that the inequality

$$|g(x)-g(y)| \leqq C\|x-y\| \tag{1}$$

holds for all x and y in B. Now any bounded uniformly continuous function on B is the uniform limit of Lipschitz functions. Let $\{p_t\}$, $t>0$, be a family of abstract Wiener measures on B. By Proposition 6 of [4], a bounded Lipschitz function g is a uniform limit of the family $\{p_t g\}$, $t\downarrow 0$. Thus, it suffices to show that for fixed positive t, the function $f=p_t g$ is of class Q^1. We consider the embedding of a Hilbert space into B which is associated with the measures $\{p_t\}$. By an earlier remark the image of the Hilbert space is dense in B. Let H denote this subspace of B. For fixed x in B and h in H the expression

$$D(x)(h) = \left[\frac{\partial}{\partial s}(f(x+sh))\right]_{s=0} \tag{2}$$

exists and defines a linear function on H by Proposition 9 of [4]. Now it is a consequence of the definition of $p_t g$ that the function f satisfies (1). Hence, from (2) we obtain the inequality

$$|D(x)(h)| \leqq C\|h\|$$

for any x in B and h in H. It follows that for fixed x, the function $D(x)$ defines an element of B^* whose B^* norm is not greater than C. We shall denote this linear functional by $\langle D(x), \cdot\rangle$.

Using equation (8) of Proposition 9 in [4], we obtain the estimate

$$|\langle D(x), h\rangle - \langle D(y), h\rangle| \leqq t^{-1/2}|h|\left\{\int_B [g(x+z)-g(y+z)]^2 p_t(dz)\right\}^{1/2}$$

for h in H, where $|h|$ is the Hilbert norm of the preimage of h. It follows from this estimate that for fixed h the function $x\to\langle D(x), h\rangle$ is continuous on B. Now since the linear functionals $D(x)$ are uniformly bounded, for any element y of B, the map $x\to\langle D(x), y\rangle$ is the uniform limit of continuous functions. Hence, the map is continuous. It then follows that the function $(x, y)\to\langle D(x), y\rangle$ is continuous on $B\times B$.

From (2) and the argument above we obtain the identity

$$f(x+y) = f(x) + \int_0^1 \langle D(x+sy), y\rangle\, ds$$

for y in H. But, each of the above expressions is a continuous function of y in the Banach space topology. Hence, the identity holds for arbitrary y in B. It follows that f is quasi-differentiable and that $f'(x) = D(x)$.

COROLLARY 1. *A real separable Banach space admits partitions of unity of class Q^1.*

PROOF. For a given Banach space let $B_r(x)$ denote an open ball of radius r centered at x in the space. It is an immediate consequence of the theorem that for any $r' < r$ there exists a function on the space of class Q^1 which vanishes outside the set $B_r(x)$ and which is equal to one on the set $B_{r'}(x)$. The existence of partitions of unity then follows from a standard argument.

COROLLARY 2. *If C_1 and C_2 are two nonempty disjoint closed subsets of a real separable Banach space then there exists a continuous quasi-differentiable function on the space which vanishes on C_1 and which is equal to one on C_2.*

PROOF. The corollary follows from an argument in [**6**, Theorem 2, p. 30].

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