ON CATEGORIES OF QUOTIENTS

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Abstract. We construct a category of quotients over a category satisfying a condition similar to the Ore condition. Addition of quotients is briefly discussed.

In a previous paper [1] a theory of relations in categories of a general type was introduced. In trying to adapt the elegant approach of Hilton [2] to the nonabelian case, a difficulty is encountered, but Hilton's method works in the associative case, in which by [1] an Ore condition holds. In this case a generalization is given which seems to be of interest; it describes a construction of a category which is algebraically a sort of category of quotients. Finally, addition is briefly discussed. Notations and definitions of [1] are freely used.

Let C be a finitely complete [3] bicategory [4] with classes of monics ∂ and epics ∂. In [1] we have constructed a near-category σC with class of objects |σC| = |C| and with relations as morphisms. We employ the method of Hilton [2] in the nonabelian case, but we have to use pairs of coinitial morphisms.

With fixed objects A, B in C consider the collection φ(A, B) of pairs of morphisms (α, β), A ↠ X ↠ B, and declare (α, β) ~ (α', β') in φ(A, B) if and only if there are σ, σ' in ∂ that satisfy ασ = α'σ', βσ = β'σ'.

In [1] we have proved that the extension σC of C is a category if and only if a condition denoted (A) holds, and that condition reads as follows: For every ξ, η in C with common codomain and with ξ, η ∈ C there is a common right multiple ξv = ηv with v ∈ C.

1. If (A) holds in C then ~ is an equivalence in φ(A, B).

For, if (ασ, βσ) = (ασ', βσ') and (ατ, βτ) = (α'τ', β'τ') with σ's and τ's in C then, by (A) and the pullback condition, there are φ, ψ ∈ C such that σφ = τψ; hence τψ, σφ ∈ C and (ασφ, βσφ) = (α'τ'ψ, β'τ'ψ).

If (A) holds we denote Z(A, B) = φ(A, B)/~. Representatives for elements of Z(A, B) can be chosen according to: (α, β) ~ (α', β') in φ(A, B) if and only if the σ-parts in i-s-factorizations of {α, β} and {α', β'}, namely {α, β}i, {α', β'}i are equivalent monics. (For, if {α, β} = {ξ, η}ξ' and {α', β'} = {ξ, η}ξ' then there are...
\( \phi, \phi' \in S \) satisfying \( \xi \phi = \xi' \phi' \), hence \( (\alpha \phi, \beta \phi) = (\alpha' \phi', \beta' \phi') \). The converse follows from the property of \( i \)-s-factorization.

We denote the equivalence class of \( (\alpha, \beta) \) by \( \beta/\alpha \). Then if \( \sigma \in S \) and \( \alpha \sigma, \beta \sigma \) are defined, we have \( \beta/\alpha = \beta \sigma/\alpha \sigma \).

For \( \beta/\alpha \in Z(A, B) \) and \( \gamma/\beta' \in Z(B, C) \) we define \( (\gamma/\beta')(\beta/\alpha) = \gamma \psi/\alpha \phi \) where \( \downarrow \psi, \phi, \beta', \beta \downarrow \) is a pullback. This composition is well defined (general proof later (IV)), it is associative and the classes \( 1/1 \) are identities.

II. \( Z \) is a category and the mapping \( \alpha \to \alpha/1 \) is a covariant embedding of \( C \) into \( Z \).

The mapping \( \alpha \to \alpha/1 \) is one-to-one: if \( \alpha/1 = \beta/1 \) then \( \alpha \sigma = \beta \sigma' \), \( 1 \sigma = 1 \sigma \) with \( \sigma, \sigma' \in S \) and, since \( S \) consists of epics only, this implies \( \alpha = \beta \).

An isomorphism between this \( Z \) and the category \( \mathfrak{D}_e \) constructed in [1] is given by assigning to \( \beta/\alpha \) of \( Z \) the relation \( [\cdot, \xi, \eta] \) with \( \{\xi, \eta\} = \{\alpha, \beta\} \).

However, this method does not enable us to construct relations in the general case. But in the \( (A) \)-case it is considerably generalized as follows.

III. Let \( C \) be a category with pullbacks (products not required). Let \( \mathfrak{D} \) be any subcategory of \( C \) with \( |\mathfrak{D}| = |C| \) and assume the following condition holds:

\( (A) \mathfrak{D}: \) If \( \downarrow v, u, \xi \downarrow \) is a pullback in \( C \) and if \( \xi \in \mathfrak{D} \) then \( v \in \mathfrak{D} \).

(We require \( A \) to hold in pullbacks since we do not assume that \( xy \in \mathfrak{D} \) implies \( x \in \mathfrak{D} \).) For \( A, B \in |C| \) denote by \( \mathfrak{D}^\phi(A, B) \) the collection of pairs \( (\alpha, \beta) \),

\[
A \leftarrow X \overset{\beta}{\rightarrow} B.
\]

We declare \( (\alpha, \beta) \sim_\mathfrak{D} (\alpha', \beta') \) in \( \mathfrak{D}^\phi(A, B) \) if there are \( \delta, \delta' \) in \( \mathfrak{D} \) satisfying \( (\alpha \delta, \beta \delta) = (\alpha' \delta', \beta' \delta') \). Then: \( \sim_\mathfrak{D} \) is an equivalence in \( \mathfrak{D}^\phi(A, B) \).

For, reflexivity follows from \( |C| = |\mathfrak{D}| \), symmetry is obvious, transitivity is similar to I.

We denote \( \mathfrak{D}^\phi(A, B) = \mathfrak{D}(A, B)/\sim_\mathfrak{D} \) and \( \beta/\alpha \) = the equivalence class of \( (\alpha, \beta) \) (more precisely \( \mathfrak{D}^\phi(A, B), \beta_\mathfrak{D} /\alpha \)).

IV. Let \( \beta/\alpha \in \mathfrak{D}^\phi(A, B) \), \( \gamma/\beta' \in \mathfrak{D}^\phi(B, C) \); if \( \downarrow \psi, \phi, \beta', \beta \downarrow \) then \( \gamma \psi/\alpha \phi \) depends only on the equivalence classes \( \beta/\alpha \) and \( \gamma/\beta' \).

**Proof.** Let \( (\alpha \delta, \beta \delta) = (\alpha_1 \delta_1, \beta_1 \delta_1) \) and \( (\beta ' \epsilon, \gamma \epsilon) = (\beta_1 ' \epsilon_1, \gamma_1 \epsilon_1) \), with \( \delta ' \) and \( \epsilon \) in \( \mathfrak{D} \). We construct pullbacks

\[
\downarrow b, a, \phi, \delta \downarrow, \quad \downarrow c, d, \psi, \epsilon \downarrow, \quad \downarrow y, x, c, b \downarrow.
\]

Then \( b, c, x, y \in \mathfrak{D} \) by \( (A)_\mathfrak{D} \). By juxtaposition of pullbacks, we obtain
The same process with the 1-indexed morphisms yields \( \downarrow dy, ax, \beta', \beta \downarrow \). By the uniqueness of pullbacks there is an invertible \( i \) that satisfies \( ax = ai_1x_1, dy = di_1y_1 \).

Therefore,

\[
(a\phi)(bx) = (ai_1\phi_1)(b_1x_1), \quad (\gamma\psi)(bx) = (ji_1\psi_1)(b_1x_1).
\]

Since \( \epsilon \in \mathcal{D} \) by \((A) \) \((\downarrow, 1, \iota^{-1}, 1 \downarrow \) and \(1 \in \mathcal{D}\)), we conclude

\[
(\alpha\phi, \gamma\psi) \sim (\alpha_1\phi_1, \gamma_1\psi_1).
\]

Now we define composition by \((\gamma/\beta')(\beta/\alpha) = \gamma\psi/\alpha\phi\) where \(\epsilon\). Theorem. \( \mathcal{E} \) with the above composition is a category with \(| \mathcal{E} | = | \mathcal{C} | \). The identity of \( \mathcal{A} \in \mathcal{E} \) in \( \mathcal{A} \in \mathcal{E} \) is \(1_1/1_1\). The mapping \( \alpha \rightarrow \alpha/1 \) of \( \mathcal{D} \) into \( \mathcal{E} \) is a covariant functor, and it is an embedding if and only if \( \mathcal{D} \) consists of epics only.

Proof. Associativity of composition follows by juxtaposition of pullbacks. The properties of the mapping \( \alpha \rightarrow \alpha/1 \) follow by using pullbacks of simple forms. The last statement: if \( \delta \in \mathcal{D} \) is not epic then there exist \( \alpha, \beta \) such that \( \alpha\delta = \beta\delta \) and \( \alpha \neq \beta \), yet \( \alpha/1 = \beta/1 \) by definition of \( \sim \). The other part of the proof is similar to II.

Denoting \( (\beta/\alpha)^{-1} = \alpha/\beta \) we have an involution on \( \mathcal{E} \) and the factorization \( \beta/\alpha = (\beta/1)(\alpha/1)^{-1} \). If all morphisms in \( \mathcal{D} \) are epic then we may identify \( \mathcal{A} \in \mathcal{E} \) with \( \alpha/1 \) in \( \mathcal{E} \), and \( \mathcal{D} \) is a sort of "category of right-quotients" over \( \mathcal{C} \) (see [1, 2.1]). Moreover, if \( \mathcal{D} \) satisfies an extra condition "\( xy \in \mathcal{D} \) and \( y \in \mathcal{D} \) imply \( x \in \mathcal{D} \)," then \( \mathcal{D} \) consists exactly of the right-regular elements, namely \( \alpha\alpha^- = 1 \) if and only if \( \alpha \in \mathcal{D} \). To prove this we first observe that for \( \alpha \in \mathcal{D} \) we have \( \alpha\alpha^- = \alpha/\alpha = 1/1 \). The converse: if \( (\alpha, \alpha) \sim (1, 1) \) then \( (\alpha\delta, \alpha\delta) = (\delta', \delta') \) with \( \delta, \delta' \in \mathcal{D} \); hence \( \alpha\delta, \delta \in \mathcal{D} \), so \( \alpha \in \mathcal{D} \).

If \( \mathcal{D} \) does not satisfy the extra condition, then there are \( x, y \) such that \( xy \in \mathcal{D} \), \( x \in \mathcal{D} \), but then by definition of \( \sim \) we have \((x, x) \sim (1, 1) \) (since \( (xy, xy) = (1xy, 1xy) \)) thus \( xx^- = 1 \) and \( x \in \mathcal{D} \).

We remark that even in the general case (nonepic \( \mathcal{D} \)) we have \( \alpha^{-} \alpha = 1 \) if and only if \( \alpha \) is monic.

If \( \mathcal{C} \) is a category with pullbacks then the subcategory of isomorphisms and \( \mathcal{C} \) itself are the two extremes for which \((A) \) holds. With the first the functor \( \alpha \rightarrow \alpha/1 \) is an embedding and only the isomorphisms of \( \mathcal{C} \) have the property \( xx^- = 1 \); even if \( x \) is a retraction we have \( xy = 1 \) but not \( xx^- = 1 \). (In the general case \( x^- \) is in the image of \( \mathcal{C} \), namely of the form \( y/1 \), iff \( x \) is a coretraction.)
The functor $C \rightarrow \mathfrak{C}_C$ is generally not an embedding. All the elements of $C$ will have the property $xx^* = 1$ in $\mathfrak{C}_C$.

An intermediate category with (A) is the category of monics in $C$ and in this case we generally do not have an embedding. The monics of $C$ have invertible images by the mapping $\alpha \mapsto \alpha/1$.

An obvious example is the following. Let $C$ be the multiplicative semigroup of positive integers (or nonzero integers). We take $D = C$ and we are in the epic case; the category $\mathfrak{C}^D$ is the group of positive (nonzero) rational numbers.

Let $G : C \rightarrow \mathfrak{G}$ be a functor and $\mathfrak{G}$ a category with an involution $(-)^*$. We ask about functors $\tilde{G}$ commuting with the involutions $G(\beta/\alpha) = \tilde{G}(\alpha/\beta)^*$ and for which the following triangle is commutative

\[
\begin{array}{ccc}
C & \rightarrow & \mathfrak{G} \\
\downarrow \tilde{G} & & \\
\mathfrak{C}^D \\
\end{array}
\]

Since $\beta/\alpha = (\beta/1)(\alpha/1)^{-1}$, we must have

$$G(\beta/\alpha) = \tilde{G}(\beta/1)\tilde{G}(\alpha/1)^* = (G\beta)(G\alpha)^*.$$ 

For $\delta \in D$ we have $\delta/\delta = 1/1$ so $(G\delta)(G\delta)^*$ must be an identity. If this is the case then $G$ is well defined since for $\beta/\alpha = \beta'/\alpha'$ we have $(\alpha\delta, \beta\delta) = (\alpha'\delta', \beta'\delta')$, so

$$\tilde{G}(\beta/\alpha) = (G\beta)(G\alpha)^* = (G\beta)(G\delta)(G\delta)^*(G\alpha)^* = G(\beta\delta)G(\alpha\delta)^* = G(\beta'\alpha')G(\alpha'\delta')^* = G(\beta'/\alpha').$$

If $\downarrow \psi, \phi, \beta, \alpha \downarrow$ then $\psi = G\phi(1/\beta)(1/\alpha) = 1/\psi$. So $(G\beta)^*G(\alpha) = \tilde{G}(1/\beta)\tilde{G}(1/\alpha) = \tilde{G}(\psi/\phi) = \tilde{G}(\psi/\phi)^{-1} = (G\psi)(G\phi)^*$. Moreover, this condition is sufficient for $\tilde{G}$ to be a functor. Given $\beta/\alpha, \gamma/\beta'$ and $\downarrow \psi, \phi, \beta', \beta \downarrow$ we have $(\gamma/\beta')(\beta/\alpha) = \gamma/\phi\alpha\phi$; hence

$$\tilde{G}[(\gamma/\beta')(\beta/\alpha)] = \tilde{G}(\gamma/\phi)G(\alpha\phi)^* = (G\gamma)(G\phi)(G\alpha)^* = (G\gamma)(G\phi)^*(G\alpha)^* = (G\gamma)(G\phi)^*(G\alpha)^* = \tilde{G}[(\gamma/\beta')(\beta/\alpha)].$$

This concludes the proof

VI. $\tilde{G}$ with the properties stated above exists if and only if $(G\delta)(G\delta)^* = 1$ for every $\delta \in D$ and $(G\beta)^*(G\alpha) = (G\psi)(G\phi)^*$ for every pullback $\downarrow \psi, \phi, \beta, \alpha \downarrow$ in $C$. 

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We say that \( \mathcal{C} \) is a **category with addition** if for some pairs of objects \( A, B \) a partial operation \( + \) is defined on \( \mathcal{C}(A, B) \) which is distributive in the following sense: if \( \alpha, \beta \in \mathcal{C}(A, B) \) and \( \alpha + \beta \) is defined, then for every \( \gamma : B \to C, \delta : D \to A \), \( \gamma \alpha + \gamma \beta \) and \( \alpha \delta + \beta \delta \) are defined and \( \gamma (\alpha + \beta) = \gamma \alpha + \gamma \beta \), \( (\alpha + \beta) \delta = \alpha \delta + \beta \delta \). (A known example of addition is the Fitting multiplication of morphisms in group theory.)

Let \( \mathcal{C} \) be a category with pullbacks and \( \mathcal{D} \) a subcategory with \( |\mathcal{D}| = |\mathcal{C}| \) and \( (A)_{\mathcal{D}} \). Let \( \beta/\alpha, \beta'/\alpha' \in (A)_{\mathcal{D}} \) and consider a pullback \( \downarrow \psi, \phi, \alpha' \), \( \alpha \downarrow \). If \( \beta \phi + \beta' \psi \) is defined in \( \mathcal{C} \) then we define

\[
\frac{\beta/\alpha + \beta'/\alpha'}{\theta} = (\beta \phi + \beta' \psi) / \theta
\]

where \( \theta = \alpha \phi = \alpha' \psi \).

For instance if \( \alpha \) is monic and \( \beta + \beta' \) is defined, then \( \beta/\alpha + \beta'/\alpha' \) is defined and \( = (\beta + \beta')/\alpha \).

**VII.** \( + \) is well defined in \( \mathcal{D} \) and it extends the addition in \( \mathcal{C} \).

**Proof.** If \( b/a = \beta/\alpha \) and \( b'/a' = \beta'/\alpha' \) then \( (a \tau, \beta \sigma) = (a \tau, \beta \sigma) \) and \( (a' \sigma', \beta' \sigma') = (a' \tau', b' \tau') \) with \( \sigma, \sigma', \tau, \tau' \in \mathcal{D} \). Now consider the following pullbacks

so by \( (A)_{\mathcal{D}} \) we have \( \beta \phi + \beta' \psi \nu \nu \) defined and

\[
\beta \phi \nu \nu + \beta' \psi \nu \nu = \beta \phi \nu \nu + \beta' \psi \nu \nu = \beta \sigma \nu \nu + \beta' \sigma' \nu \nu \nu'
\]

\[
= (b \tau) \nu \nu + (b' \tau') \nu \nu' \nu'.
\]

By the extended definition of \( + \) and, considering the big pullback, we have \( b \tau / a \tau + b' \tau' / a' \tau' \) defined and \( = (b \phi \nu \nu + b' \psi \nu \nu') / \eta \) where \( \eta = \alpha \sigma \nu \nu = \alpha' \sigma' \nu \nu \nu \). But \( \tau, \tau' \in \mathcal{D} \), hence \( b \tau / a \tau = b / a \) and \( b' \tau' / a' \tau' = b' / a' \), so, since \( w, v \in \mathcal{D} \), we conclude

\[
b/a + b'/a' = (b \phi \nu \nu + b' \psi \nu \nu') / \alpha \phi \nu \nu = (b \phi + b' \psi) \nu \nu / \alpha \phi \nu \nu
\]

\[
= (b \phi + b' \psi) / \alpha \phi = \beta / \alpha + \beta'/\alpha'.
\]

The fact \( \alpha/1 + \alpha'/1 = (\alpha + \alpha')/1 \) is obvious.

As Hilton pointed out [2], even in the abelian case and with \( \mathcal{D} \) the
class of epics, the extended addition is not distributive. At least with \( \delta \in \mathbb{C} \) we have

\[
(\delta/1)(\beta/\alpha + \beta'/\alpha') = (\delta/1)(\beta/\alpha) + (\delta/1)(\beta'/\alpha'),
\]

since, with the notation above,

\[
(\delta/1)(\beta/\alpha + \beta'/\alpha') = (\delta/1)(\beta \Phi + \beta' \Psi)/\alpha \Phi = \delta(\beta \Phi + \beta' \Psi)/\alpha \Phi
\]

\[
= (\delta \beta \Phi + \delta \beta' \Psi)/\alpha \Phi = \delta \beta/\alpha + \delta \beta'/\alpha'
\]

\[
= (\delta/1)(\beta/\alpha) + (\delta/1)(\beta'/\alpha').
\]

Let us compare

\[
(\delta/\gamma)(\beta/\alpha + \beta'/\alpha'), \quad (\delta/\gamma)(\beta/\alpha) + (\delta/\gamma)(\beta'/\alpha')
\]
in the general case. Again \( \beta/\alpha + \beta'/\alpha' = (\beta \Phi + \beta' \Psi)/\theta \). Let

\[
\downarrow \xi', \xi, \gamma, \beta \Phi + \beta' \Psi \downarrow
\]

and then the left side is \( \delta \xi'/\theta \xi \). To compute the right-hand side, let \( \downarrow s, t, \gamma, \beta \downarrow \) and \( \downarrow s', t', \gamma, \beta' \downarrow \); and then we have to construct the sum \( \delta s/\alpha t + \delta s'/\alpha t' \). But here we encounter a question of existence since, having \( \downarrow x', x, \alpha t', \alpha t \), we need the sum \( \delta x + \delta x' \) in \( \mathbb{C} \). We have \( \alpha x = \alpha t' x' \); hence there is a \( \lambda \) satisfying \( \alpha x = \phi \lambda, t' x' = \psi \lambda \). By our assumptions \( \beta \Phi \lambda + \beta' \Psi \lambda = \beta t x + \beta' t' x' = \gamma s x + \gamma s' x' \) is defined. Here we need an additional assumption, and this case could be, for instance, the assumption of Kurosh et al. [5] that for \( \gamma \) monic, if \( \gamma f + \gamma g \) is defined then \( f + g \) is defined. Assuming this we have \( \delta x + \delta x' \) defined, provided that \( \gamma \) is monic. Then, by \( (\beta \Phi + \beta' \Psi) \lambda = \gamma (s x + s' x') \), there is a \( \mu \) for which \( \xi \mu = \lambda, \xi' \mu = s x + s' x' \), hence

\[
(\delta/\gamma)(\beta/\alpha) + (\delta/\gamma)(\beta'/\alpha') = \delta \xi' \mu/\alpha \Phi \xi \mu,
\]

whereas \( (\delta/\gamma)(\beta/\alpha + \beta'/\alpha') = \delta \xi'/\alpha \Phi \xi \). Unfortunately, in the general case \( \mu \in \mathbb{D} \), otherwise we would have at least left distributivity for \( \delta/\gamma \) with \( \gamma \) monic.

Let us compare

\[
(\beta/\alpha + \beta'/\alpha')(\delta/\gamma), \quad (\beta/\alpha)(\delta/\gamma) + (\beta'/\alpha')(\delta/\gamma).
\]

Assume that the right-hand side is defined; so if \( \downarrow v, u, \delta, \alpha \downarrow \), then \( (\beta/\alpha)(\delta/\gamma) \beta u/\gamma v \) and similarly \( (\beta'/\alpha')(\delta/\gamma) = \beta u'/\gamma v' \), thus with \( \downarrow y, \gamma, \gamma v, \gamma v \downarrow \) we have \( \beta u y + \beta' u' y' \) defined in \( \mathbb{C} \). If the left side is defined, then with \( \downarrow \psi, \phi, \alpha', \alpha \downarrow \) we have \( \beta/\alpha + \beta' \alpha' = (\beta \Phi + \beta' \Psi) / \theta \), \( \theta = \alpha \Phi = \alpha' \Psi \). Therefore, with \( \downarrow \eta, \xi, u, \phi \downarrow \), \( (\beta/\alpha + \beta'/\alpha)(\delta/\gamma) = (\beta \Phi + \beta' \Psi) \xi/\gamma v \eta \). Since \( \alpha \Phi = \alpha' \Psi \), there is a pullback \( \downarrow \eta', \xi, u', \psi \downarrow \)
such that $v'\eta' = v\eta$. So $\gamma v\eta = \gamma v'\eta'$ and a $\mu$ exists such that $\gamma\mu = \eta$, $y'\mu = \eta'$, hence

$$(\beta/\alpha + \beta'/\alpha')(\delta/\gamma) = (\beta w\eta + \beta' w'\eta')/\gamma v\eta = (\beta w\gamma + \beta' w'\gamma')\mu/\gamma v\eta$$

and $\gamma v\eta = (\gamma v\gamma)\mu = (\gamma v'\gamma')\mu$. Again $\mu$ is not necessarily in $\mathcal{D}$, otherwise we could have right distributivity.

VIII. In the particular case $\mathcal{D} = \mathcal{C}$, assuming that the involved sums are defined, we have both right- and left-distributivity.

IX. In the bicategorical case ($\mathcal{D} = \mathcal{S}$), assuming that the involved sums are defined, we have at least inequalities

$$[S][R] + [R'] \geq [S][R] + [S][R'];$$

$$[R][T] + [R'][T] \geq ([R] + [R'])[T],$$

for relations in $\mathcal{O}_e$.

REFERENCES


