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ON CATEGORIES OF QUOTIENTS

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ABSTRACT. We construct a category of quotients over a category satisfying a condition similar to the Ore condition. Addition of quotients is briefly discussed.

In a previous paper [1] a theory of relations in categories of a general type was introduced. In trying to adapt the elegant approach of Hilton [2] to the nonabelian case, a difficulty is encountered, but Hilton's method works in the associative case, in which by [1] an Ore condition holds. In this case a generalization is given which seems to be of interest; it describes a construction of a category which is algebraically a sort of category of quotients. Finally, addition is briefly discussed. Notations and definitions of [1] are freely used.

Let $\mathcal{C}$ be a finitely complete [3] bicategory [4] with classes of monics $\mathcal{I}$ and epics $\mathcal{S}$. In [1] we have constructed a near-category $\mathcal{R}_{\mathcal{C}}$ with class of objects $|\mathcal{R}_{\mathcal{C}}| = |\mathcal{C}|$ and with relations as morphisms. We employ the method of Hilton [2] in the nonabelian case, but we have to use pairs of coinitial morphisms.

With fixed objects A, B in $\mathcal{C}$ consider the collection $\mathcal{P}(A, B)$ of pairs of morphisms (α, β), $A \xleftarrow{\alpha} X \xrightarrow{\beta} B$, and declare $(\alpha, \beta) \sim (\alpha', \beta')$ in $\mathcal{P}(A, B)$ if and only if there are σ, σ' in $\mathcal{S}$ that satisfy $\alpha\sigma = \alpha'\sigma'$, $\beta\sigma = \beta'\sigma'$.

In [1] we have proved that the extension $\mathcal{R}_{\mathcal{C}}$ of $\mathcal{C}$ is a category if and only if a condition denoted (**A**) holds, and that condition reads as follows: For every ξ, η in $\mathcal{C}$ with common codomain and with $\xi \in \mathcal{S}$ there is a common right multiple $\xi u = \eta v$ with $v \in \mathcal{S}$.

I. *If* (**A**) *holds in* $\mathcal{C}$ *then* $\sim$ *is an equivalence in* $\mathcal{P}(A, B)$.

For, if $(\alpha\sigma, \beta\sigma) = (\alpha'\sigma', \beta'\sigma')$ and $(\alpha'\tau, \beta'\tau) = (\alpha''\tau', \beta''\tau')$ with σ's and τ's in $\mathcal{S}$ then, by (**A**) and the pullback condition, there are $\phi, \psi \in \mathcal{S}$ such that $\sigma'\phi = \tau\psi$; hence $\tau'\psi, \sigma\phi \in \mathcal{S}$ and $(\alpha\sigma\phi, \beta\sigma\phi) = (\alpha''\tau'\psi, \beta''\tau'\psi)$.

If (**A**) holds we denote $\mathcal{Z}(A, B) = \mathcal{P}(A, B)/\sim$. Representatives for elements of $\mathcal{Z}(A, B)$ can be chosen according to: $(\alpha, \beta) \sim (\alpha', \beta')$ in $\mathcal{P}(A, B)$ if and only if the $\mathcal{I}$-parts in i-s-factorizations of $\{\alpha, \beta\}$ and $\{\alpha', \beta'\}$, namely $\{\alpha, \beta\}^i$, $\{\alpha', \beta'\}^i$ are equivalent monics. (For, if $\{\alpha, \beta\} = \{\xi, \eta\}\zeta$ and $\{\alpha', \beta'\} = \{\xi, \eta\}\zeta'$ with $\zeta, \zeta' \in \mathcal{S}$ then there are

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$\phi, \phi' \in \mathcal{S}$ satisfying $\zeta\phi = \zeta'\phi'$, hence $(\alpha\phi, \beta\phi) = (\alpha'\phi', \beta'\phi')$. The converse follows from the property of i-s-factorization.)

We denote the equivalence class of (α, β) by β/α. Then if $\sigma \in \mathcal{S}$ and $\alpha\sigma$, $\beta\sigma$ are defined, we have $\beta/\alpha = \beta\sigma/\alpha\sigma$.

For $\beta/\alpha \in \mathcal{Z}(A, B)$ and $\gamma/\beta' \in \mathcal{Z}(B, C)$ we define $(\gamma/\beta')(\beta/\alpha) = \gamma\psi/\alpha\phi$ where $\downarrow \psi, \phi, \beta', \beta \downarrow$ is a pullback. This composition is well defined (general proof later (IV)), it is associative and the classes $1/1$ are identities.

II. *$\mathcal{Z}$ is a category and the mapping $\alpha \to \alpha/1$ is a covariant embedding of $\mathcal{C}$ into $\mathcal{Z}$.*

The mapping $\alpha \to \alpha/1$ is one-to-one: if $\alpha/1 = \beta/1$ then $\alpha\sigma = \beta\sigma'$, $1\sigma = 1\sigma'$ with $\sigma, \sigma' \in \mathcal{S}$ and, since $\mathcal{S}$ consists of epics only, this implies $\alpha = \beta$.

An isomorphism between this $\mathcal{Z}$ and the category $\mathcal{R}_{\mathcal{C}}$ constructed in [1] is given by assigning to β/α of $\mathcal{Z}$ the relation $[\cdot, \xi, \eta]$ with $\{\xi, \eta\} = \{\alpha, \beta\}^i$.

However, this method does not enable us to construct relations in the general case. But in the (**A**)-case it is considerably generalized as follows.

III. Let $\mathcal{C}$ be a category with pullbacks (products not required). Let $\mathcal{D}$ be any subcategory of $\mathcal{C}$ with $|\mathcal{D}| = |\mathcal{C}|$ and assume the following condition holds:

$(\mathbf{A})_{\mathcal{D}}$: *If $\downarrow v, u, \eta, \xi \downarrow$ is a pullback in $\mathcal{C}$ and if $\xi \in \mathcal{D}$ then $v \in \mathcal{D}$.* (We require (**A**) to hold in pullbacks since we do not assume that $xy \in \mathcal{D}$ implies $x \in \mathcal{D}$.) For $A, B \in |\mathcal{C}|$ denote by $\mathcal{P}^{\mathcal{D}}(A, B)$ the collection of pairs (α, β),

$$A \xleftarrow{\alpha} X \xrightarrow{\beta} B.$$

We declare $(\alpha, \beta) \sim_{\mathcal{D}} (\alpha', \beta')$ in $\mathcal{P}^{\mathcal{D}}(A, B)$ if there are δ, δ' in $\mathcal{D}$ satisfying $(\alpha\delta, \beta\delta) = (\alpha'\delta', \beta'\delta')$. Then: *$\sim_{\mathcal{D}}$ is an equivalence* in $\mathcal{P}^{\mathcal{D}}(A, B)$. For, reflexivity follows from $|\mathcal{C}| = |\mathcal{D}|$, symmetry is obvious, transitivity is similar to I.

We denote $\mathcal{R}^{\mathcal{D}}(A, B) = \mathcal{P}^{\mathcal{D}}(A, B)/\sim_{\mathcal{D}}$ and $\beta/\alpha =$ the equivalence class of (α, β) (more precisely $\mathcal{R}^{\mathcal{D}}_{\mathcal{C}}(A, B)$, $\beta/_{\mathcal{D}}\,\alpha$).

IV. *Let $\beta/\alpha \in \mathcal{R}^{\mathcal{D}}(A, B)$, $\gamma/\beta' \in \mathcal{R}^{\mathcal{D}}(B, C)$; if $\downarrow \psi, \phi, \beta', \beta \downarrow$ then $\gamma\psi/\alpha\phi$ depends only on the equivalence classes β/α and γ/β'.*

PROOF. Let $(\alpha\delta, \beta\delta) = (\alpha_1\delta_1, \beta_1\delta_1)$ and $(\beta'\epsilon, \gamma\epsilon) = (\beta_1'\epsilon_1, \gamma_1\epsilon_1)$, with δ's and ϵ's in $\mathcal{D}$. We construct pullbacks

$$\downarrow b, a, \phi, \delta \downarrow, \qquad \downarrow c, d, \psi, \epsilon \downarrow, \qquad \downarrow y, x, c, b \downarrow .$$

Then $b, c, x, y \in \mathcal{D}$ by $(\mathbf{A})_{\mathcal{D}}$. By juxtaposition of pullbacks, we obtain

$\downarrow dy, ax, \beta'\epsilon, \beta\delta \downarrow$. The same process with the 1-indexed morphisms yields $\downarrow d_1y_1, a_1x_1, \beta_1'\epsilon_1, \beta_1\delta_1 \downarrow$ and $b_1, c_1, x_1, y_1 \in \mathfrak{D}$. By the uniqueness of pullbacks there is an invertible ι that satisfies $ax = a_1x_1\iota$, $dy = d_1y_1\iota$. Therefore,

$$(\alpha\phi)(bx) = (\alpha_1\phi_1)(b_1x_1\iota), \qquad (\gamma\psi)(bx) = (\gamma_1\psi_1)(b_1x_1\iota).$$

Since $\iota \in \mathfrak{D}$ by $(\mathbf{A})_{\mathfrak{D}}$ $(\downarrow \iota, 1, \iota^{-1}, 1 \downarrow$ and $1 \in \mathfrak{D})$, we conclude

$$(\alpha\phi, \gamma\psi) \sim (\alpha_1\phi_1, \gamma_1\psi_1).$$

Now we define composition by $(\gamma/\beta')(\beta/\alpha) = \gamma\psi/\alpha\phi$ where $\downarrow \psi, \phi, \beta', \beta \downarrow$.

V. Theorem. *$\mathfrak{R}_{\mathfrak{C}}^{\mathfrak{D}}$ with the above composition is a category with $|\mathfrak{R}_{\mathfrak{C}}^{\mathfrak{D}}| = |\mathfrak{C}|$. The identity of $A \in |\mathfrak{C}|$ in $\mathfrak{R}_{\mathfrak{C}}^{\mathfrak{D}}$ is $1_A/1_A$. The mapping $\alpha \to \alpha/1$ of $\mathfrak{C}$ into $\mathfrak{R}_{\mathfrak{C}}^{\mathfrak{D}}$ is a covariant functor, and it is an embedding if and only if $\mathfrak{D}$ consists of epics only.*

Proof. Associativity of composition follows by juxtaposition of pullbacks. The properties of the mapping $\alpha \to \alpha/1$ follow by using pullbacks of simple forms. The last statement: if $\delta \in \mathfrak{D}$ is not epic then there exist α, β such that $\alpha\delta = \beta\delta$ and $\alpha \neq \beta$, yet $\alpha/1 = \beta/1$ by definition of $\sim_{\mathfrak{D}}$. The other part of the proof is similar to II.

Denoting $(\beta/\alpha)^- = \alpha/\beta$ we have an involution on $\mathfrak{R}^{\mathfrak{D}}$ and the factorization $\beta/\alpha = (\beta/1)(\alpha/1)^-$. If all morphisms in $\mathfrak{D}$ are epic then we may identify $\alpha \in \mathfrak{C}$ with $\alpha/1$ in $\mathfrak{R}^{\mathfrak{D}}$, and $\mathfrak{R}^{\mathfrak{D}}$ is a sort of "*category of right-quotients*" over $\mathfrak{C}$ (see [1, 2.1]). Moreover, if $\mathfrak{D}$ satisfies an extra condition "*$xy \in \mathfrak{D}$ and $y \in \mathfrak{D}$ imply $x \in \mathfrak{D}$*," then $\mathfrak{D}$ consists exactly of the right-regular elements, namely $\alpha\alpha^- = 1$ if and only if $\alpha \in \mathfrak{D}$. To prove this we first observe that for $\alpha \in \mathfrak{D}$ we have $\alpha\alpha^- = \alpha/\alpha = 1/1$. The converse: if $(\alpha, \alpha) \sim (1, 1)$ then $(\alpha\delta, \alpha\delta) = (\delta', \delta')$ with $\delta, \delta' \in \mathfrak{D}$; hence $\alpha\delta, \delta \in \mathfrak{D}$, so $\alpha \in \mathfrak{D}$.

If $\mathfrak{D}$ does not satisfy the extra condition, then there are x, y such that $xy, y \in \mathfrak{D}$, $x \notin \mathfrak{D}$, but then by definition of $\sim_{\mathfrak{D}}$ we have $(x, x) \sim (1, 1)$ (since $(xy, xy) = (1xy, 1xy)$) thus $xx^- = 1$ and $x \notin \mathfrak{D}$.

We remark that even in the general case (nonepic $\mathfrak{D}$) we have $\alpha^-\alpha = 1$ if and only if α is monic.

If $\mathfrak{C}$ is a category with pullbacks then the subcategory of isomorphisms and $\mathfrak{C}$ itself are the two extremes for which $(\mathbf{A})$ holds. With the first the functor $\alpha \to \alpha/1$ is an embedding and only the isomorphisms of $\mathfrak{C}$ have the property $xx^- = 1$; even if x is a retraction we have $xy = 1$ but not $xx^- = 1$. (In the general case x^- is in the image of $\mathfrak{C}$, namely of the form $y/1$, iff x is a coretraction.)

The functor $\mathcal{C} \to \mathcal{R}^{\mathcal{C}}_{\mathcal{C}}$ is generally not an embedding. All the elements of $\mathcal{C}$ will have the property $xx^{-}=1$ in $\mathcal{R}^{\mathcal{C}}_{\mathcal{C}}$.

An intermediate category with (**A**) is the category of monics in $\mathcal{C}$ and in this case we generally do not have an embedding. The monics of $\mathcal{C}$ have invertible images by the mapping $\alpha \to \alpha/1$.

An obvious example is the following. Let $\mathcal{C}$ be the multiplicative semigroup of positive integers (or nonzero integers). We take $\mathcal{D}=\mathcal{C}$ and we are in the epic case; the category $\mathcal{R}^{\mathcal{D}}$ is the group of positive (nonzero) rational numbers.

Let $G:\mathcal{C} \to \mathcal{A}$ be a functor and $\mathcal{A}$ a category with an involution $(-)^*$. We ask about functors $\tilde{G}$ commuting with the involutions $(\tilde{G}(\beta/\alpha)=\tilde{G}(\alpha/\beta)^*)$ and for which the following triangle is commutative

$$\begin{tikzcd} \mathcal{C} \arrow[r, "G"] \arrow[d] & \mathcal{A} \\ \mathcal{R}^{\mathcal{D}} \arrow[ur, "\tilde{G}"'] & \end{tikzcd}$$

Since $\beta/\alpha=(\beta/1)(\alpha/1)^{-}$, we must have

$$\tilde{G}(\beta/\alpha) = \tilde{G}(\beta/1)\tilde{G}(\alpha/1)^* = (G\beta)(G\alpha)^*.$$

For $\delta \in \mathcal{D}$ we have $\delta/\delta=1/1$ so $(G\delta)(G\delta)^*$ must be an identity. If this is the case then $\tilde{G}$ is well defined since for $\beta/\alpha=\beta'/\alpha'$ we have $(\alpha\delta, \beta\delta)=(\alpha'\delta', \beta'\delta')$, so

$$\begin{aligned} \tilde{G}(\beta/\alpha) &= (G\beta)(G\alpha)^* = (G\beta)(G\delta)(G\delta)^*(G\alpha)^* = G(\beta\delta)G(\alpha\delta)^* \\ &= G(\beta'\delta')G(\alpha'\delta')^* = G(\beta'/\alpha'). \end{aligned}$$

If $\downarrow \psi, \phi, \beta, \alpha \downarrow$ then $(1/\beta)(\alpha/1)=\psi/\phi$. So

$$(G\beta)^*G(\alpha) = \tilde{G}(1/\beta)\tilde{G}(\alpha/1) = \tilde{G}(\psi/\phi) = \tilde{G}((\psi/1)(\phi/1)^{-}) = (G\psi)(G\phi)^*.$$

Moreover, this condition is sufficient for $\tilde{G}$ to be a functor. Given β/α, γ/β' and $\downarrow \psi, \phi, \beta', \beta \downarrow$ we have $(\gamma/\beta')(\beta/\alpha)=\gamma\psi/\alpha\phi$; hence

$$\begin{aligned} \tilde{G}[(\gamma/\beta')(\beta/\alpha)] &= \tilde{G}(\gamma\psi/\alpha\phi) = G(\gamma\psi)G(\alpha\phi)^* = (G\gamma)(G\psi)(G\phi)^*(G\alpha)^* \\ &= (G\gamma)(G\beta')^*(G\beta)(G\alpha)^* = \tilde{G}(\gamma/\beta')\tilde{G}(\beta/\alpha). \end{aligned}$$

This concludes the proof of

VI. *$\tilde{G}$ with the properties stated above exists if and only if $(G\delta)(G\delta)^*=1$ for every $\delta \in \mathcal{D}$ and $(G\beta)^*(G\alpha)=(G\psi)(G\phi)^*$ for every pullback $\downarrow \psi, \phi, \beta, \alpha \downarrow$ in $\mathcal{C}$.*

We say that $\mathcal{C}$ is a *category with addition* if for some pairs of objects A, B a partial operation $+$ is defined on $\mathcal{C}(A, B)$ which is distributive in the following sense: if $\alpha, \beta \in \mathcal{C}(A, B)$ and $\alpha+\beta$ is defined, then for every $\gamma: B \to C$, $\delta: D \to A$, $\gamma\alpha+\gamma\beta$ and $\alpha\delta+\beta\delta$ are defined and $\gamma(\alpha+\beta) = \gamma\alpha+\gamma\beta$, $(\alpha+\beta)\delta = \alpha\delta+\beta\delta$. (A known example of addition is the Fitting multiplication of morphisms in group theory.)

Let $\mathcal{C}$ be a category with pullbacks and $\mathfrak{D}$ a subcategory with $|\mathfrak{D}| = |\mathcal{C}|$ and $(\mathbf{A})_{\mathfrak{D}}$. Let $\beta/\alpha, \beta'/\alpha' \in \mathcal{R}^{\mathfrak{D}}(A, B)$ and consider a pullback $\downarrow \psi, \phi, \alpha', \alpha \downarrow$. If $\beta\phi+\beta'\psi$ is defined in $\mathcal{C}$ then we define

$$\beta/\alpha + \beta'/\alpha' = (\beta\phi + \beta'\psi)/\theta$$

where $\theta = \alpha\phi = \alpha'\psi$.

For instance if α is monic and $\beta+\beta'$ is defined, then $\beta/\alpha+\beta'/\alpha'$ is defined and $= (\beta+\beta')/\alpha$.

VII. $+$ *is well defined in* $\mathcal{R}^{\mathfrak{D}}$ *and it extends the addition in* $\mathcal{C}$.

PROOF. If $b/a = \beta/\alpha$ and $b'/a' = \beta'/\alpha'$ then $(\alpha\sigma, \beta\sigma) = (a\tau, b\tau)$ and $(\alpha'\sigma', \beta'\sigma') = (a'\tau', b'\tau')$ with $\sigma, \sigma'\tau, \tau' \in \mathfrak{D}$. Now consider the following pullbacks

$$\begin{tikzcd} \cdot \arrow[r, "w"] \arrow[d, "w'"'] & \cdot \arrow[r, "u"] \arrow[d, "v"'] & \cdot \arrow[d, "\sigma"] \\ \cdot \arrow[r, "v'"'] \arrow[d, "u'"'] & \cdot \arrow[r, "\phi"'] \arrow[d, "\psi"'] & \cdot \arrow[d, "\alpha"] \\ \cdot \arrow[r, "\sigma'"'] & \cdot \arrow[r, "\alpha'"'] & A \end{tikzcd}$$

so by $(\mathbf{A})_{\mathfrak{D}}$ we have $\beta\phi+\beta'\psi vw$ defined and

$$\begin{aligned} \beta\phi vw + \beta'\psi vw &= \beta\phi vw + \beta'\psi v'w' = \beta\sigma uw + \beta'\sigma'u'w' \\ &= (b\tau)uw + (b'\tau')u'w'. \end{aligned}$$

By the extended definition of $+$ and, considering the big pullback, we have $b\tau/a\tau+b'\tau'/a'\tau'$ defined and $= (\beta\phi vw+\beta'\psi v'w')/\eta$ where $\eta = \alpha\sigma uw = \alpha'\sigma'u'w'$. But $\tau, \tau' \in \mathfrak{D}$, hence $b\tau/a\tau = b/a$ and $b'\tau'/a'\tau' = b'/a'$, so, since $w, v \in \mathfrak{D}$, we conclude

$$\begin{aligned} b/a + b'/a' &= (\beta\phi vw + \beta'\psi vw)/\alpha\phi vw = (\beta\phi + \beta'\psi)vw/\alpha\phi vw \\ &= (\beta\phi + \beta'\psi)/\alpha\phi = \beta/\alpha + \beta'/\alpha'. \end{aligned}$$

The fact $\alpha/1+\alpha'/1 = (\alpha+\alpha')/1$ is obvious.

As Hilton pointed out [2], even in the abelian case and with $\mathfrak{D}$ the class of epics, the extended addition is not distributive. At least with $\delta \in \mathfrak{C}$ we have

$$(\delta/1)(\beta/\alpha + \beta'/\alpha') = (\delta/1)(\beta/\alpha) + (\delta/1)(\beta'/\alpha'),$$

since, with the notation above,

$$\begin{aligned}(\delta/1)(\beta/\alpha + \beta'/\alpha') &= (\delta/1)(\beta\phi + \beta'\psi)/\alpha\phi = \delta(\beta\phi + \beta'\psi)/\alpha\phi \\ &= (\delta\beta\phi + \delta\beta'\psi)/\alpha\phi = \delta\beta/\alpha + \delta\beta'/\alpha' \\ &= (\delta/1)(\beta/\alpha) + (\delta/1)(\beta'/\alpha').\end{aligned}$$

Let us compare

$$(\delta/\gamma)(\beta/\alpha + \beta'/\alpha'), \qquad (\delta/\gamma)(\beta/\alpha) + (\delta/\gamma)(\beta'/\alpha')$$

in the general case. Again $\beta/\alpha+\beta'/\alpha' = (\beta\phi+\beta'\psi)/\theta$. Let

$$\downarrow \xi', \xi, \gamma, \beta\phi + \beta'\psi \downarrow$$

and then the left side is $\delta\xi'/\theta\xi$. To compute the right-hand side, let $\downarrow s, t, \gamma, \beta \downarrow$ and $\downarrow s', t', \gamma, \beta' \downarrow$; and then we have to construct the sum $\delta s/\alpha t+\delta s'/\alpha' t'$. But here we encounter a question of existence since, having $\downarrow x', x, \alpha' t', \alpha t \downarrow$, we need the sum $\delta sx+\delta s'x'$ in $\mathfrak{C}$. We have $\alpha tx = \alpha' t' x'$; hence there is a λ satisfying $tx = \phi\lambda$, $t'x' = \psi\lambda$. By our assumptions $\beta\phi\lambda+\beta'\psi\lambda = \beta tx+\beta' t' x' = \gamma sx+\gamma s' x'$ is defined. Here we need an additional assumption, and this case could be, for instance, the assumption of Kurosh et al. [5] that for γ monic, if $\gamma f+\gamma g$ is defined then $f+g$ is defined. Assuming this we have $\delta sx+\delta s'x'$ defined, provided that γ is monic. Then, by $(\beta\phi+\beta'\psi)\lambda = \gamma(sx+s'x')$, there is a μ for which $\xi\mu = \lambda$, $\xi'\mu = sx+s'x'$, hence

$$(\delta/\gamma)(\beta/\alpha) + (\delta/\gamma)(\beta'/\alpha') = \delta\xi'\mu/\alpha\phi\xi\mu,$$

whereas $(\delta/\gamma)(\beta/\alpha+\beta'/\alpha') = \delta\xi'/\alpha\phi\xi$. Unfortunately, in the general case $\mu \notin \mathfrak{D}$, otherwise we would have at least left distributivity for δ/γ with γ monic.

Let us compare

$$(\beta/\alpha + \beta'/\alpha')(\delta/\gamma), \qquad (\beta/\alpha)(\delta/\gamma) + (\beta'/\alpha')(\delta/\gamma).$$

Assume that the right-hand side is defined; so if $\downarrow v, u, \delta, \alpha \downarrow$, then $(\beta/\alpha)(\delta/\gamma) = \beta u/\gamma v$ and similarly $(\beta'/\alpha')(\delta/\gamma) = \beta u'/\gamma v'$, thus with $\downarrow y', y, \gamma v', \gamma v \downarrow$ we have $\beta uy+\beta' u' y'$ defined in $\mathfrak{C}$. If the left side is defined, then with $\downarrow \psi, \phi, \alpha', \alpha \downarrow$ we have $\beta/\alpha+\beta'\alpha' = (\beta\phi+\beta'\psi)/\theta$, $\theta = \alpha\phi = \alpha'\psi$. Therefore, with $\downarrow \eta, \xi, u, \phi \downarrow$, $(\beta/\alpha+\beta'/\alpha)(\delta/\gamma) = (\beta\phi+\beta'\psi)\xi/\gamma v\eta$. Since $\alpha\phi = \alpha'\psi$, there is a pullback $\downarrow \eta', \xi, u', \psi \downarrow$

such that $v'\eta' = v\eta$. So $\gamma v\eta = \gamma v'\eta'$ and a μ exists such that $y\mu = \eta$, $y'\mu = \eta'$, hence

$$(\beta/\alpha + \beta'/\alpha')(\delta/\gamma) = (\beta u\eta + \beta' u'\eta')/\gamma v\eta = (\beta uy + \beta' u'y')\mu/\gamma v\eta$$

and $\gamma v\eta = (\gamma vy)\mu = (\gamma v'y')\mu$. Again μ is not necessarily in $\mathfrak{D}$, otherwise we could have right distributivity.

VIII. In the particular case $\mathfrak{D} = \mathfrak{C}$, assuming that the involved sums are defined, we have both right- and left-distributivity.

IX. In the bicategorical case ($\mathfrak{D} = \mathfrak{S}$), assuming that the involved sums are defined, we have at least inequalities

$$[S]([R] + [R']) \geqq [S][R] + [S][R'];$$

$$[R][T] + [R'][T] \geqq ([R] + [R'])[T],$$

for relations in $\mathfrak{R}_{\mathfrak{C}}$.

References

1. A. Klein, *Relations in categories*, Illinois J. Math. **14** (1970), 536–550.

2. P. Hilton, *Correspondences and exact squares*, Proc. Conference on Categorical Algebra (La Jolla, Calif., 1965), Springer, New York, 1966, pp. 254–271. MR **34** #4326.

3. B. Mitchell, *Theory of categories*, Pure and Appl. Math., vol. **17**, Academic Press, New York, 1965. MR **34** #2647.

4. J. Isbell, *Natural sums and abelianizing*, Pacific J. Math. **14** (1964), 1265–1281. MR **31** #3478.

5. A. G. Kuroš, A. H. Livšic and E. G. Šul'geĭfer, *Foundations of the theory of categories*, Uspehi Mat. Nauk **15** (1960), no. 6 (96), 3–52 = Russian Math. Surveys **15** (1960), 1–46. MR **23** #A1688; MR **22** #9526.

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