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TWO REMARKS ON THE GROUP ALGEBRA OF A FINITE GROUP

K. L. FIELDS[1]

ABSTRACT. If $K\subseteq Q(\zeta_m)$, m least, we find the smallest n such that $M_n(K)$ appears in QG for some finite group G when m is either a prime power or not exactly divisible by a prime to the first power. We also show that every group of even order possesses a nontrivial real valued character of Schur index 1 over the rationals.

1. It is well known that if $M_n(K)$, the algebra of $n\times n$ matrices over the field K, appears as a simple component in the rational group algebra QG of a finite group, then $K\subseteq Q(\zeta_m)$ for some m; conversely, given $K\subseteq Q(\zeta_m)$ for some m, $M_n(K)$ appears in some QG for some n. We ask: What is the least such n? We can answer when $K\subseteq Q(\zeta_{p^a})$, i.e., when m is a power of a prime:

Let $Q\subseteq K\subseteq Q(\zeta_{p^a})$, and assume that a is the smallest exponent of p which suffices. It is clear that if $M_n(K)$ occurs in QG then G contains an element of order p^a. Now if $A^m=1$, m least, for some $A\in M_n(K)$, then $\phi(m)\leqq n[K:Q]$ (since the dimension over Q of every maximal subfield of $M_n(K)$ equals $n[K:Q]$). In particular, n must be at least $[Q(\zeta_{p^a}):K]$. We claim that this value of n suffices: for let $G_0=\text{gal}(Q(\zeta_{p^a})/K)$ and $G=\langle x\rangle \times_{sd} G_0$ where $x^{p^a}=1$ and G_0 acts on x in the same way as it acts on ζ_{p^a}. Then QG contains as a simple component the crossed product $\langle Q(\zeta_{p^a}),\ G_0,\ 1\rangle\cong M_n(K)$ where $n=[Q(\zeta_{p^a}):K]$.

By Corollary 3 of Brauer [2], this argument also shows that n_{minimal} is $[Q(\zeta_m):K]$ whenever m is not exactly divisible by any prime to the first power.

2. It is well known that every group of even order possesses a nontrivial real valued character. We extend this to the following:

THEOREM. *Every group of even order possesses a nontrivial real valued absolutely irreducible character whose Schur index over the rationals is* 1, *which generates a field of odd degree over the rationals, and is of odd degree.*

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PROOF. $|G| = \sum m_Q(\chi)^2[Q(\chi):Q][\chi(1)/m_Q(\chi)]^2+1$ where the summation is over the nontrivial, nonconjugate, absolutely irreducible characters of G. Since $|G|$ is even, $[Q(\chi):Q]$ must be odd for at least one χ. By the Brauer-Speiser Theorem [1], $m_Q(\chi)$ is either 1 or 2 for all such χ, and so there must exist a nontrivial character ν (in fact, an odd number of them) such that $[Q(\nu):Q]$ is odd, $m_Q(\nu)=1$, and $\nu(1)$ is odd.

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UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS 60637