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ON THE UNIVALENCE OF SOME CLASSES OF REGULAR FUNCTIONS

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ABSTRACT. Let $F(z)$ be regular in the unit disk $\Delta=\{z:|z|<1\}$ and normalized by the conditions $F(0)=0$ and $F'(0)=1$ and let $2f(z)=[zF(z)]'$. The paper deals with the mapping properties of $f(z)$ when $F(z)$ is known. For example, if $F(z)$ is starlike of order α, $0\leqq\alpha<1$, then the disk in which $f(z)$ is always starlike of order β, $\alpha\leqq\beta<1$, is determined. All results are sharp.

1. Introduction. Let $\mathcal{S}$ denote the class of functions $f(z)$ regular and univalent in the open unit disk $\Delta=\{z:|z|<1\}$ which are normalized by the conditions $f(0)=0$ and $f'(0)=1$. Furthermore, let $\mathcal{K}$, $\mathcal{S}^*$ and $\mathcal{C}$ denote the subclasses of $\mathcal{S}$ consisting of convex, starlike and close-to-convex functions; then, as is well known, $\mathcal{K}\subset\mathcal{S}^*\subset\mathcal{C}$. Recently, Libera [2] showed that if $f(z)$ is in $\mathcal{K}$, $\mathcal{S}^*$ or $\mathcal{C}$, then $F(z)=(2/z)\int_0^z f(\zeta)d\zeta$ is in $\mathcal{K}$, $\mathcal{S}^*$ or $\mathcal{C}$, respectively. On the other hand Livingston [3] has studied the converse question, i.e., he has studied the mapping properties of the function $f(z)$ defined by

$$f(z) = \tfrac{1}{2}(zF(z))', \tag{1.1}$$

when $F(z)$ is in one of the above subclasses of $\mathcal{S}$ (and $'$ denotes differentiation with respect to z). For example, he has proved that if $F(z)$ is in $\mathcal{S}^*$, then $f(z)$, given by (1.1), is starlike for $|z|<\frac{1}{2}$ and, in general, in no larger disk centered at the origin.

More recently Padmanabhan [5] has refined the results of Livingston by imposing further restrictions on the character of $F(z)$. A normalized, regular and univalent function $F(z)$ is starlike of order α, i.e., $F(z)\in\mathcal{S}^*(\alpha)$, for $0\leqq\alpha<1$, if and only if $\operatorname{Re}\{zF'(z)/F(z)\}>\alpha$ for z in Δ. His main theorem shows that if $F(z)\in\mathcal{S}^*(\alpha)$, for $0\leqq\alpha\leqq\frac{1}{2}$, then $f(z)$, in (1.1) is starlike of the same order α, for $|z|<\{\alpha-2+(\alpha^2+4)^{1/2}\}/2\alpha$. He obtains analogous results when $F(z)$ is convex of order α in Δ, written $F(z)\in\mathcal{K}(\alpha)$; when $F(z)$ is in $\mathcal{C}(\alpha,\beta)$, i.e., "close-to-convex of order α and type β" in Δ as defined by Libera [1]; or in the case when $\operatorname{Re}\{F'(z)\}>\alpha$ for z in Δ and $0\leqq\alpha<1$.

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The purpose of the present note is to extend and generalize the results of Padmanabhan in the following ways. His main theorem is extended to include the range of α when $\frac{1}{2}<\alpha<1$ and generalized by finding the sharp radius of the disk in which $\operatorname{Re}\{zf'(z)/f(z)\}>\beta$ when $F(z)$ is in $\mathcal{S}^*(\alpha)$, $0\leqq\alpha<1$, $0\leqq\beta<1$ and $\beta\geqq\alpha$. Also, if $\operatorname{Re}\{F'(z)\}>\alpha$, z in Δ, then the sharp radius of the disk for which $\operatorname{Re}\{f'(z)\}>\beta$ is given explicitly for arbitrary α and β in the interval $[0, 1)$.

2. Theorems and their proofs.

THEOREM 1. *If $f(z)$ is in $\mathcal{S}^*(\alpha)$ for $0\leqq\alpha<1$, $f(z)=\frac{1}{2}(zF(z))'$ with z in Δ and $\alpha\leqq\beta<1$, then $\operatorname{Re}\{zf'(z)/f(z)\}>\beta$ for $|z|<r_0$, where r_0 is the smallest positive root of the equation*

$$(2.1)\quad (1-\beta)+(2(2\alpha-1)-\beta(1+\alpha))r+\alpha(2\alpha-\beta-1)r^2=0.$$

This result cannot be improved.

Before giving the proof of this theorem we show, by specializing choices of α and β, how this result implies some of the work of Padmanabhan [**5**].

COROLLARY 1. *If $0\leqq\alpha<1$ and $F(z)$ is in $\mathcal{S}^*(\alpha)$, then $f(z)$ is starlike of order α for*

$$(2.2)\qquad |z|<\{\alpha-2+(\alpha^2+4)^{1/2}\}/2\alpha$$

and this bound is sharp.

This corollary extends the fundamental theorem of Padmanabhan beyond the range $0\leqq\alpha\leqq\frac{1}{2}$ and is obtained by setting $\alpha=\beta$ in Theorem 1. (Theorems 2 and 3, by Padmanabhan, can be extended in a similar fashion, since these are corollaries to his Theorem 1.)

By choosing $\alpha=0$ in Theorem 1, above, we get the following new result.

COROLLARY 2. *If $F(z)$ is starlike in Δ, $F(z)\in\mathcal{S}^*$, then $f(z)$ is starlike of order β for $|z|<(1-\beta)/(2+\beta)$, which is sharp.*

Corollaries 1 and 2 give the earlier result of Livingston [**3**] when $\alpha=\beta=0$.

We turn now to the proof of Theorem 1. $F(z)$ is in $\mathcal{S}^*(\alpha)$ if and only if $\operatorname{Re}\{zF'(z)/F(z)\}>\alpha$ for z in Δ. Consequently, there is a function $\omega(z)$ satisfying Schwarz's lemma such that

$$(2.3)\qquad \frac{zF'(z)}{F(z)}=\frac{1+(2\alpha-1)\omega(z)}{1+\omega(z)},\qquad z\in\Delta.$$

The definition of $f(z)$, as in (1.1), yields

$$\frac{zf(z) - \int_0^z f(\zeta)d\zeta}{\int_0^z f(\zeta)d\zeta} = \frac{zF'(z)}{F(z)}, \qquad \text{for } z \text{ in } \Delta. \tag{2.4}$$

Equating these we have

$$f(z) = \frac{2(1+\alpha\omega(z))}{z(1+\omega(z))}\int_0^z f(\zeta)d\zeta = \left\{\frac{1+\alpha\omega(z)}{1+\omega(z)}\right\}F(z) \tag{2.5}$$

and a differentiation gives

$$\frac{zf'(z)}{f(z)} = \frac{\alpha z\omega'(z)}{1+\alpha\omega(z)} - \frac{z\omega'(z)}{1+\omega(z)} + \frac{zF'(z)}{F(z)}, \tag{2.6}$$

which together with (2.3) becomes

$$\frac{zf'(z)}{f(z)} = \frac{1+(2\alpha-1)\omega(z) - z\omega'(z)}{1+\omega(z)} + \frac{\alpha z\omega'(z)}{1+\alpha\omega(z)}; \tag{2.7}$$

and

$$\begin{aligned}
\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \beta\right\}
&= \operatorname{Re}\left\{\frac{\begin{array}{l}[1+(2\alpha-1)\omega(z)-z\omega'(z)](1+\alpha\omega(z)) \\ \qquad +\alpha z\omega'(z)(1+\omega(z))-\beta(1+\omega(z))(1+\alpha\omega(z))\end{array}}{(1+\omega(z))(1+\alpha\omega(z))}\right\} \\
&= \frac{|1+\alpha\omega(z)|^2[2(\alpha-\beta)\operatorname{Re}\{\omega(z)\}+(2\alpha-\beta-1)|\omega(z)|^2+(1-\beta)]}{|1+\omega(z)|^2|1+\alpha\omega(z)|^2} \\
&\quad - \frac{(1-\alpha)\operatorname{Re}\{z\omega'(z)(1+\overline{\omega(z)})(1+\alpha\overline{\omega(z)})\}}{|1+\omega(z)|^2|1+\alpha\omega(z)|^2} \\
&= \frac{|1+\alpha\omega(z)|^2[(\alpha-\beta)|1+\omega(z)|^2+(1-\alpha)(1-|\omega(z)|^2)]}{|1+\omega(z)|^2|1+\alpha\omega(z)|^2} \\
&\quad - \frac{(1-\alpha)\operatorname{Re}\{z\omega'(z)(1+\overline{\omega(z)})(1+\alpha\overline{\omega(z)})\}}{|1+\omega(z)|^2|1+\alpha\omega(z)|^2}.
\end{aligned} \tag{2.8}$$

Now, $\operatorname{Re}\{zf'(z)/f(z)\} > \beta$ only if the numerator of the last quotient in (2.8) is positive and this is implied by

$$(2.9)\quad \begin{aligned} &|1+\alpha\omega(z)|^2[(\alpha-\beta)|1+\omega(z)|^2+(1-\alpha)(1-|\omega(z)|^2)] \\ &\qquad -(1-\alpha)|z\omega'(z)|\cdot|1+\omega(z)|\cdot|1+\alpha\omega(z)| > 0. \end{aligned}$$

Using the bound $|\omega'(z)| \leqq (1-|\omega(z)|^2)/(1-r^2)$, $|z|=r$, (see [**4**, p. 168]), and dividing by the positive factor $|1+\alpha\omega(z)|\cdot|1+\omega(z)|$, we see that (2.9) holds whenever

$$(2.10)\quad \begin{aligned} &\left|\frac{1+\alpha\omega(z)}{1+\omega(z)}\right|[(\alpha-\beta)|1+\omega(z)|^2+(1-\alpha)(1-|\omega(z)|^2)] \\ &\qquad -(1-\alpha)\frac{r(1-|\omega(z)|^2)}{1-r^2} > 0. \end{aligned}$$

Making use of our assumption that $\alpha \leqq \beta$ and using the relation $|1+\omega(z)| \leqq 1+|\omega(z)|$ we see (2.10) is satisfied if

$$(2.11)\quad \begin{aligned} &\left|\frac{1+\alpha\omega(z)}{1+\omega(z)}\right|[(1+|\omega(z)|)((1-\beta)+(2\alpha-\beta-1)|\omega(z)|)] \\ &\qquad -\frac{(1-\alpha)r(1-|\omega(z)|^2)}{1-r^2} > 0. \end{aligned}$$

Let $Q(r)$ denote the quadratic defined in (2.1), let r_0 be its smallest positive zero and restrict r so that $r \leqq r_0$. $Q(0)=(1-\beta)>0$ and $Q((1-\beta)/(1+\beta-2\alpha))<0$, therefore $0<r_0<(1-\beta)/(1+\beta-2\alpha)$; and since $|\omega(z)| \leqq r$, $(1-\beta)+(2\alpha-\beta-1)|\omega(z)|>0$. Now, multiplying (2.11) by $(1-r^2)/(1+|\omega(z)|)$ and making use of the inequality $|(1+\alpha\omega(z))/(1+\omega(z))| \geqq (1+\alpha r)/(1+r)$ we see that (2.11) holds true whenever

$$(2.12)\quad \begin{aligned} &(1+\alpha r)(1-r)[(1-\beta)+(2\alpha-\beta-1)|\omega(z)|] \\ &\qquad -(1-\alpha)r(1-|\omega(z)|) > 0, \end{aligned}$$

or

$$(2.13)\quad \begin{aligned} &[(1-\beta)+(1-\alpha)(\beta-2)r-\alpha(1-\beta)r^2] \\ &+[(2\alpha-\beta-1)+(1-\alpha)(2+\beta-2\alpha)r-\alpha(2\alpha-\beta-1)r^2] \\ &\qquad \cdot|\omega(z)| > 0. \end{aligned}$$

Let $P(r)$ represent the coefficient of $|\omega(z)|$ in (2.13) and r_1 be its smallest positive zero. $P(0)<0$, therefore $P(r)<0$ for $0 \leqq r<r_1$. We wish to show that $r_0 \leqq r_1$. $P(r)+Q(r)=2(\alpha-\beta)(1+2\alpha r) \leqq 0$ for all r in the interval $[0, 1]$, hence we have, in particular, that $Q(r_1)=P(r_1)+Q(r_1) \leqq 0$ and, therefore, $r_0 \leqq r_1$. Consequently, $P(r)<0$ for $0<r \leqq r_0$, and because $|\omega(z)| \leqq r$, (2.13) is implied by

$$\begin{aligned}(1-\beta)+(4\alpha-\alpha\beta-3)+(2+\beta-5\alpha+2\alpha^2)r^2 \\ +\alpha(1+\beta-2\alpha)r^3 \\ =(1-r)[(1-\beta)+(2(2\alpha-1)-\beta(1+\alpha))r \\ +\alpha(2\alpha-\beta-1)r^2]\geqq 0.\end{aligned} \tag{2.14}$$

The relation in (2.14) is valid whenever $r<r_0$. This gives the first part of Theorem 1.

To show these results are sharp for all admissible α and β we need only replace $\omega(z)$ by z in (2.8) and obtain the quadratic (2.1) in the numerator of the second term of (2.8). This term is zero at r_0.

The authors were not able to obtain suitable results for the complementary case when $\beta<\alpha$ by the above and other similar methods.

The remainder of this note deals with the case when $F'(z)$ has a suitably restricted and positive real part in Δ. To simplify the presentation we introduce the class $\mathcal{P}$ and prove two lemmas relating to this class. $P(z)$ is in $\mathcal{P}$ if and only if $P(z)$ is regular and $\operatorname{Re}\{P(z)\}>0$ for z in Δ and $P(0)=1$.

LEMMA 1. *For μ real and $|z|=r$, $0\leqq r<1$,*

$$\begin{aligned}2\operatorname{Re}\left\{\frac{1+e^{i\mu}z}{1-e^{i\mu}z}+\frac{e^{i\mu}z}{(1-e^{i\mu}z)^2}\right\} \\ \geqq \frac{2(1-r-r^2)}{(1+r)^2}, \qquad \textit{if } r\leqq\sqrt{7}-2, \\ \geqq -\frac{(1-3r^2)^2}{4(1-r^2)^2}, \qquad \textit{if } \sqrt{7}-2<r<1.\end{aligned} \tag{2.15}$$

PROOF. Let $z=re^{i\phi}$, then

$$\begin{aligned}\operatorname{Re}\left\{\frac{1+e^{i\mu}z}{1-e^{i\mu}z}+\frac{e^{i\mu}z}{(1-e^{i\mu}z)^2}\right\} \\ =\operatorname{Re}\left\{\frac{1+e^{i(\mu+\phi)}r}{1-e^{i(\mu+\phi)}r}+\frac{e^{i(\mu+\phi)}r}{(1-e^{i(\mu+\phi)}r)^2}\right\} \\ =\operatorname{Re}\left\{\frac{(1+e^{i(\mu+\phi)}r)(1-e^{i(\mu+\phi)}r)+e^{i(\mu+\phi)}r}{(1-e^{i(\mu+\phi)}r)^2}\right\} \\ =\frac{(1-2r^2-r^4)+(3r^3-r)\cos(\mu+\phi)}{(1-2r\cos(\mu+\phi)+r^2)^2}=H(\phi).\end{aligned} \tag{2.16}$$

Therefore for fixed r, $0\leqq r<1$, and fixed μ, we seek to minimize $H(\phi)$.

A differentiation shows that $H'(\phi)=0$ if and only if ϕ is $-\mu$, $\pi-\mu$ or ϕ_1 where $\cos(\mu+\phi_1)=[3-6r^2-r^4]/[2r(1-3r^2)]$. The relation defining ϕ_1 has meaning only when $r>\sqrt{7}-2$, otherwise the magnitude of the defining expression exceeds 1.

Consequently, if $0\leqq r\leqq\sqrt{7}-2$, then the minimum of $H(\phi)$ is either $H(-\mu)$ or $H(\pi-\mu)$; a brief calculation shows it is $H(\pi-\mu)$ and the value $H(\pi-\mu)$ appears in the lemma.

On the other hand if r exceeds $\sqrt{7}-2$, then the minimum is either $H(\pi-\mu)$ or $H(\phi_1)$.

$$\begin{aligned} H(\phi_1)-H(\pi-\mu) &= \frac{-(1-3r^2)^2}{4(1-r^2)^2}-\frac{2(1-r-r^2)}{(1+r)^2} \\ &= \frac{-(r^4+8r^3+10r^2-24r+9)}{(1+r)^2(1-r)^2} \qquad (2.17)\\ &= -\left[\frac{(r-(\sqrt{7}-2))(r+(\sqrt{7}+2))}{(1+r)(1-r)}\right]^2 \leqq 0. \end{aligned}$$

Therefore $H(\phi_1)$ is the minimum when $\sqrt{7}-2<r<1$; $H(\phi_1)$ is the appropriate value appearing in (2.15).

Lemma 2. *For* $|z|=r$, $0<r<1$,

$$\begin{aligned} \min_{P(z)\in\mathcal{P}} \operatorname{Re}\{2P(z)+zP'(z)\} & \\ &= \frac{2(1-r-r^2)}{(1+r)^2}, \qquad \textit{if } r\leqq\sqrt{7}-2, \qquad (2.18)\\ &= -\frac{(1-3r^2)^2}{4(1-r^2)^2}, \qquad \textit{if } \sqrt{7}-2<r<1. \end{aligned}$$

These results are sharp.

Proof. If $P(z)\in\mathcal{P}$, then by the well-known Herglotz-Stieltjes representation [6] we may write

$$P(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1+e^{i\theta}z}{1-e^{i\theta}z}\,d\alpha(\theta),$$

where $\alpha(\theta)$ is real-valued and nondecreasing in $[0,2\pi]$ and $\int_0^{2\pi}d\alpha(\theta)=2\pi$. From this it follows that

$$\operatorname{Re}\{2P(z)+zP'(z)\}=\frac{1}{\pi}\int_0^{2\pi}\operatorname{Re}\left\{\frac{1+e^{i\theta}z}{1-e^{i\theta}z}+\frac{e^{i\theta}z}{(1-e^{i\theta}z)^2}\right\}d\alpha(\theta).$$

The inequalities in (2.18) follow immediately from Lemma 1.

The bounds given in Lemma 2 are rendered sharp at $z=r$ by a function of the form

$$P(z) = \frac{1+\rho}{2}\left(\frac{1+e^{i\theta}z}{1-e^{i\theta}z}\right) + \frac{1-\rho}{2}\left(\frac{1+e^{-i\theta}z}{1-e^{-i\theta}z}\right);$$

where $\rho=1$ and $\theta=0$ for the case $0 \leqq r \leqq \sqrt{7}-2$; and

$$\theta = \text{arc}\cos[3 - 6r^2 - r^4]/[2r(1 - 3r^2)]$$

and ρ is arbitrary, $-1 \leqq \rho \leqq 1$, for the case $\sqrt{7}-2 < r < 1$.

We now proceed to apply Lemma 2 in the proof of the following theorem.

THEOREM 2. *For* $0 \leqq \alpha, \beta < 1$ *and* $0 \leqq r < 1$ *let*

$$N(r) = (1 - \beta) + (3\alpha - 2\beta - 1)r + (2\alpha - \beta - 1)r^2 \tag{2.19}$$

and

$$M(r) = (9\alpha - 8\beta - 1) + (6 - 22\alpha + 16\beta)r^2 + (17\alpha - 8\beta - 9)r^4. \tag{2.20}$$

If $F(z)$ *and* $f(z)$ *are related as in* (1.1) *and* $\text{Re}\{F'(z)\} > \alpha$, $z \in \Delta$, *then* $\text{Re}\{f'(z)\} > \beta$ *for* $|z| < r_0$, *where* r_0 *is the smallest positive root of the equation*

(2.21) $N(r)=0$, *when* $\beta \geqq \alpha$, *or* $\beta < \alpha$ *and* $\alpha \leqq (8-3\sqrt{7})/(16-5\sqrt{7})$, *or* $((16 - 5\sqrt{7})\alpha + 3(\sqrt{7} - 8))/(8 - 2\sqrt{7}) \leqq \beta < \alpha$ *and* $(8-3\sqrt{7})/(16-5\sqrt{7}) < \alpha$;

and r_0 *is the root greater than and closest to* $(\sqrt{7}-2)$ *of the equation* $M(r) = 0$, *when*

$$\beta < ((16-5\sqrt{7})\alpha + (3\sqrt{7}-8))/(8-2\sqrt{7}) \text{ and } \alpha > (8-3\sqrt{7})/(16-5\sqrt{7}).$$

These results are sharp.

Choosing $\alpha=\beta$ in Theorem 2 gives as a special case, Padmanabhan's Theorem 4, [5], stated below.

COROLLARY 3. *If* $0 \leqq \alpha < 1$, $\text{Re}\{F'(z)\} > \alpha$, z *in* Δ, *and* $f(z)$ *is defined as in* (1.1), *then* $\{\text{Re } f'(z)\} > \alpha$ *for* $|z| < (\sqrt{5}-1)/2$, *and this cannot be improved.*

This corollary is obtained from (2.19) in which case

$$N(r) = (1 - \alpha)(1 - r - r^2).$$

In the case $\alpha=0$, (2.19) has the form

$$N(r) = (1-\beta) - (2\beta+1)r - (1+\beta)r^2;$$

from which we get the following.

COROLLARY 4. *If $f(z)$ and $F(z)$ are as in* (1.1) *and $F'(z) \in \mathcal{P}$, then* $\operatorname{Re}\{f'(z)\} > \beta$, $0 \leqq \beta < 1$, *for* $|z| < ((4\beta+5)^{1/2} - (2\beta+1))/2(1+\beta)$ *and in general in no larger disk.*

Both these corollaries reduce to a result of Livingston [3] when $\alpha = \beta = 0$. The case $\beta = 0$, and α arbitrary yields another interesting but somewhat more cumbersome case. We now give a proof of Theorem 2.

Since $\operatorname{Re}\{F'(z)\} > \alpha$ there is a function $Q(z)$ in $\mathcal{P}$ such that $F'(z) = (1-\alpha)Q(z) + \alpha$ for z in Δ. Using this and (1.1) we may write

$$2\operatorname{Re}\{f'(z) - \beta\} = (1-\alpha)\operatorname{Re}\{2Q(z) + zQ'(z)\} + 2(\alpha - \beta). \tag{2.22}$$

If $|z| = r \leqq \sqrt{7} - 2$, it follows from the lemma that

$$\begin{aligned}\operatorname{Re}\{f'(z) - \beta\} &> (1-\alpha)\left(\frac{1-r-r^2}{(1+r)^2}\right) + (\alpha-\beta)\\ &= \frac{(1-\beta) + (3\alpha - 2\beta - 1)r + (2\alpha - \beta - 1)r^2}{(1+r)^2} = \frac{N(r)}{(1+r)^2}.\end{aligned} \tag{2.23}$$

For $\beta \geqq \alpha$, $N(r) = (1-\beta)(1-r-r^2) + (\alpha-\beta)(3r+2r^2)$ and $N((\sqrt{5}-1)/2) \leqq 0$. Because $(\sqrt{5}-1)/2 < \sqrt{7}-2$, the smallest positive root r_0 of $N(r) = 0$ is less than $\sqrt{7}-2$, hence $\operatorname{Re}\{f'(z)\} > \beta$ for $|z| < r_0$.

If $\beta < \alpha$ and $\alpha \leqq (8 - 3\sqrt{7})/(16 - 5\sqrt{7})$, then

$$\begin{aligned}N(\sqrt{7}-2) &= \beta(2\sqrt{7}-8) + (16-5\sqrt{7})\alpha + (3\sqrt{7}-8)\\ &\leqq \beta(2\sqrt{7}-8) \leqq 0.\end{aligned}$$

Thus, again in this case, the solution to our problem is given by the smallest positive root of $N(r) = 0$.

If

$$[(16 - 5\sqrt{7})\alpha + (3\sqrt{7} - 8)]/(8 - 2\sqrt{7}) \leqq \beta < \alpha$$

and

$$(8 - 3\sqrt{7})/(16 - 5\sqrt{7}) < \alpha,$$

then

$$N(\sqrt{7}-2) = \beta(2\sqrt{7}-8) + (16-5\sqrt{7})\alpha + (3\sqrt{7}-8) < 0.$$

Consequently, as in the above cases, $\operatorname{Re}\{f'(z)\}>\beta$ for $|z|<r_0$, where r_0 is the smallest positive root of $N(r)=0$.

Assuming $\alpha>(8-3\sqrt{7})/(16-5\sqrt{7})$ and

$$\beta<[(16-5\sqrt{7})\alpha+(3\sqrt{7}-8)]/(8-2\sqrt{7}),$$

we have

$$(1-\beta)>\frac{(5\sqrt{7}-10)\alpha+(16-5\sqrt{7})}{8-2\sqrt{7}}, \tag{2.24}$$

$$(3\alpha-2\beta-1)>\frac{(2\sqrt{7}-4)\alpha+(4-2\sqrt{7})}{4-\sqrt{7}} \tag{2.25}$$

and

$$(2\alpha-\beta-1)>\frac{\sqrt{7}(\alpha-1)}{8-2\sqrt{7}}; \tag{2.26}$$

and these imply that

$$N(r)>\frac{(1-r)}{8-2\sqrt{7}}[(16-5\sqrt{7})+2(2-\sqrt{7})r-\sqrt{7}r^2]>0$$

$$\text{for } r\leqq\sqrt{7}-2.$$

Therefore $\operatorname{Re}\{f'(z)\}>\beta$ for $r\leqq\sqrt{7}-2$, in this case; and we conclude from (2.22) and the lemma that, for $\sqrt{7}-2<r<1$,

$$\begin{aligned} 2\operatorname{Re}\{f'(z)-\beta\} &\geqq \frac{-(1-\alpha)(1-3r^2)^2}{4(1-r^2)^2}+2(\alpha-\beta) \\ &= \frac{(9\alpha-8\beta-1)+(6-22\alpha+16\beta)r^2+(17\alpha-8\beta-9)r^4}{4(1-r^2)^2} \\ &= \frac{M(r)}{4(1-r^2)^2}. \end{aligned} \tag{2.27}$$

Since the two bounds given in (2.15) agree for $r=\sqrt{7}-2$, then necessarily $M(\sqrt{7}-2)>0$. It is easily checked that $M(1)<0$. Therefore $M(r)$ has at least one root r_0 such that $(\sqrt{7}-2)<r_0<1$. If we choose r_0 to be the root closest to $\sqrt{7}-2$, then $\operatorname{Re}\{f'(z)\}>\beta$ for $|z|<r_0$.

The statement about sharpness in the theorem follows, since the bounds given in the lemma are sharp.

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