PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 30, Number 2, October 1971

THE TOTAL SYMBOL OF A PSEUDO-DIFFERENTIAL OPERATOR

BENT E. PETERSEN

ABSTRACT. This note presents a direct completely coordinate-free proof that the symbols of a pseudo-differential operator are differential operators whose coefficients are functions on the higher order cotangent bundles.

Let M be a smooth manifold and let E and F be complex smooth vector bundles on M (smooth $=C^\infty$). In [1] L. Hörmander defines a *pseudo-differential operator* from E to F to be a continuous linear map $P:\Gamma_c^\infty(E)\to\Gamma^\infty(F)$ such that there exists a sequence $r_0>r_1>\cdots\to-\infty$ of real numbers such that we have an asymptotic expansion

$$e^{-i\lambda g}P(e^{i\lambda g}s)\sim\sum_{k=0}^{\infty}P_k(s,g)\lambda^{r_k} \tag{1}$$

in the following sense. If $s\in\Gamma_c^\infty(E)$ and if K is a compact subset of $C^\infty(M:R)$ such that $g\in K$ implies $dg\neq 0$ on supp s then, for each integer $N>0$,

$$\left\{\lambda^{-r_N}\left(e^{-i\lambda g}P(e^{i\lambda g}s)-\sum_{k<N}P_k(s,g)\lambda^{r_k}\right):\lambda\geqq 1,\ g\in K\right\} \tag{2}$$

is a bounded subset of $\Gamma^\infty(F)$. Here $\Gamma^\infty(F)$ denotes the smooth sections of F, $\Gamma_c^\infty(E)$ the compactly supported smooth sections of E, and $C^\infty(M:R)$ the smooth real-valued functions on M. All these spaces are provided with the usual Schwartz topologies.

A few properties of the asymptotic expansion are immediate. From (2) we see that, for each integer $n\geqq 0$,

$$P_n(s,g)=\lim_{\lambda\to\infty}\lambda^{-r_n}\left(e^{-i\lambda g}P(e^{i\lambda g}s)-\sum_{k<n}P_k(s,g)\lambda^{r_k}\right) \tag{3}$$

where the convergence is in the topology of $\Gamma^\infty(F)$, uniformly for $g\in K$. By induction it follows that $P_n(s,g)$ depends continuously on g for $g\in K$. From (3) we also have that the expansion (1) is unique and that $P_n(s,g)$ is positively homogeneous in g of degree r_k, and linear in s for supp $s\subseteq\{x\in M:dg(x)\neq 0\}$. Now suppose A is a com-

Received by the editors February 1, 1971.
AMS 1970 *subject classifications*. Primary 58G15, 35S05.
Key words and phrases. Pseudo-differential operators, symbol.

Copyright © 1971, American Mathematical Society

pact subset of M and let K be a compact subset of $C^\infty(M:R)$ such that $g\in K$ implies $dg\neq 0$ on A. The subspace $\Gamma^\infty(E|A)$ of $\Gamma_c^\infty(E)$ consisting of sections with supports in A is a Fréchet space. Then by the Banach-Steinhaus theorem and induction on (3), $s\to P_n(s, g)$ is a continuous linear map of $\Gamma^\infty(E|A)$ into $\Gamma^\infty(F)$ for each $g\in K$ and we have the following stronger version of (2). If B is a bounded subset of $\Gamma_c^\infty(E)$ and K is a compact subset of $C^\infty(M:R)$ and $c>0$, such that if $g\in K$ and $s\in B$ then $|dg|\geqq c$ on supp s, then, for each integer $N>0$,

$$(2)' \qquad \left\{\lambda^{-r_N}\left(e^{-i\lambda g}P(e^{i\lambda g}s) - \sum_{k<N} P_k(s, g)\lambda^{r_k}\right):\lambda \geqq 1,\, g\in K,\, s\in B\right\}$$

is a bounded subset of $\Gamma^\infty(F)$.

The purpose of this note is to give a completely coordinate-free proof of the following theorem.

THEOREM. *Let P be a pseudo-differential operator with asymptotic expansion* (1). *For each integer $k\geqq 0$ there exists a unique fibre-preserving smooth map*

$$\sigma_k(P): T_0^{*n_k+1} \to \operatorname{Hom}(J^{n_k}(E), F)$$

positively homogeneous of degree r_k, where n_k is the greatest integer in r_0-r_k, such that if $s\in\Gamma_c^\infty(E)$, $g\in C^\infty(M:R)$, $dg\neq 0$ on supp s, *then*

$$\sigma_k(P)\cdot j_{n_k}(dg)\cdot j_{n_k}(s) = P_k(s, g).$$

In particular

$$\sigma_0(P): T^* - (0) \to \operatorname{Hom}(E, F) \quad \textit{and} \quad \sigma_0(P)\cdot dg\cdot s = P_0(s, g).$$

This theorem has previously been announced in [**3**] and [**4**] without proof. $\sigma_k(P)$ is called the kth *symbol* of P and $\sigma_0(P)$ is called the *top order symbol* or frequently just the *symbol* of P. A proof of the theorem may be based on the explicit coordinate expressions given in [**1**] for the P_k. These expressions however follow from some delicate Fourier transform estimates whereas the present method involves only some simple manipulations of asymptotic series.

We begin by explaining some of the notation. $J^n(E)$ denotes the nth jet bundle of E [**2**, Chapter 4], and j_n is the nth jet extension map. T^{*n} is the nth order cotangent bundle of M and may conveniently be defined as the kernel of the canonical morphism $J^n(1)\to 1$ where 1 denotes the trivial *real* line bundle. It is easy to see that T^{*n} may be regarded as a subbundle of $J^{n-1}(T^*)$ and that if $g\in C^\infty(M:R)$ then $j_{n-1}(dg)$ is a section of T^{*n} and that such sections generate the fibres. For each $n\geqq 1$ we have a natural morphism $\pi_n: T^{*n}\to T^*$ and we now define $T_0^{*n}=\pi_n^{-1}(T^*-(0))$. In particular $T_0^{*1}=T^*-(0)$. Here $T^*-(0)$ denotes the complement of the zero section in T^*.

The theorem then says that $P_k(s, g)$ is a linear differential operator of degree $\leqq n_k$ in s with coefficients that are (nonlinear) differential operators in dg of degree $\leqq n_k$. At the end of the proof we will show that whereas the dependence on dg may be highly nonlinear, the derivatives of dg enter polynomially. The following lemma is the key to the proof. We note that by adding some zero terms to the expansion (1) we may assume that for each integer $k\geqq 0$ there exists an integer $k+>k$ such that $r_{k+}=r_k-1$.

LEMMA. *Let $s\in\Gamma_c^\infty(E)$, $g, h\in C^\infty(M:R)$ and suppose that $d(g+th)\neq 0$ on* supp *s for $0\leqq t\leqq 1$. Then for each $x\in M$ we have*

$$(4)\quad P_k(s, g+h)(x) - P_k(s, g)(x) = i\int_0^1 P_{k+}((h-h(x))s, g+th)(x)dt.$$

To prove the lemma note that

$$\lambda^{-r_N}\left(e^{-i\lambda(g+th)}P(e^{i\lambda(g+th)}(h-h(x))s) - \sum_{k<N} P_k((h-h(x))s, g+th)\lambda^{r_k}\right)$$

when evaluated at x is a bounded set in $C([0, 1], F_x)$ for $\lambda\geqq 1$. If we integrate over t we obtain a bounded set in F_x, i.e. we have an asymptotic expansion

$$e^{-i\lambda g(x)}\int_0^1 P(e^{i\lambda g}(h-h(x))e^{i\lambda t(h-h(x))}s)(x)dt$$
$$\sim\sum_{k=0}^\infty \lambda^{r_k}\int_0^1 P_k((h-h(x))s, g+th)(x)dt.$$

But now

$$e^{i\lambda g}(e^{i\lambda(h-h(x))}-1)s = i\lambda\int_0^1 e^{i\lambda g}(h-h(x))e^{i\lambda t(h-h(x))}s\,dt$$

where the integral converges in $\Gamma_c^\infty(E)$, since the integrand as a function of t belongs to $C([0, 1], \Gamma_c^\infty(E))$. Thus P commutes with the integral and therefore

$$e^{-i\lambda(g+h)(x)}P(e^{i\lambda(g+h)}s)(x) - e^{-i\lambda g(x)}P(e^{i\lambda g}s)(x)$$
$$= e^{-i\lambda g}P(e^{i\lambda g}(e^{i\lambda(h-h(x))}-1)s)(x)$$
$$\sim\sum_{k=0}^\infty \lambda^{r_k+1}i\int_0^1 P_k((h-h(x))s, g+th)(x)dt$$

which completes the proof of the Lemma. We now prove the Theorem.

Suppose $s\in\Gamma_c^\infty(E)$, $g\in C^\infty(M:R)$ and $x\in M$. Suppose $h\in C^\infty(M:R)$, $h(x)=0$ and $d(g+th)\neq 0$ on supp s for $0\leqq t\leqq 1$. Replacing h by rh and t by tr^{-1} $(0<r<1)$ in (4) by continuity of the argument of the integral we see that

$$iP_{k+}(hs,\, g)(x) = (d/dt)P_k(s,\, g+th)(x)\,\big|_{t=0}.$$

Putting it differently, if for each integer $k\geqq 0$ we define $P_{k-}(s, g)$ by

$$\begin{aligned} P_{k-}(s,\, g) &= P_l(s,\, g) && \text{if } l+ = k \quad \text{for some } l\geqq 0,\\ &= 0 && \text{otherwise,} \end{aligned}$$

then

$$iP_k(hs,\, g)(x) = (d/dt)P_{k-}(s,\, g+th)(x)\,\big|_{t=0} \tag{5}$$

where (5) may be seen to hold for those k not of the form $l+$ merely by adding sufficient zero terms to the expansion (1) before going through the above arguments.

Suppose now that we have shown that P_{k-} is a differential operator in s of degree $\leqq N$. If s vanishes at x of order $N+1$ then $P_{k-}(s, g+th)(x)=0$ and hence, by (5), $P_k(hs, g)(x)=0$. Thus P_k is a differential operator in s of degree $\leqq N+1$. Since $P_{k-}=0$ if $r_0-r_k<1$ it follows by induction that P_k is a differential operator in s of degree $\leqq n_k$. In particular P_k is local in s.

Now suppose that $s\in\Gamma_c^\infty(E)$, g, $g'\in C^\infty(M:R)$ with $dg\neq 0$ and $dg'\neq 0$ on supp s and suppose that for some $x\in M$ we have $j_{n_k}(dg)(x)=j_{n_k}(dg')(x)$. We wish to show that $P_k(s, g)(x)=P_k(s, g')(x)$. If $x\notin$ supp s we are done since P_k is local in s. If $x\in$supp s since $dg(x)=dg'(x)$ and P_k is local in s we may cut down the support of s so that $(1-t)dg+tdg'\neq 0$ on supp s for $0\leqq t\leqq 1$. Now let $h=g-g'$ so $h-h(x)$ vanishes of order at least $n_k+2=n_{k+}+1$ at x. Then

$$P_{k+}((h-h(x))s,\, g+th)(x) = 0$$

for $0\leqq t\leqq 1$, since P_{k+} is a differential operator in s of degree $\leqq n_{k+}$. Thus by (4) we have

$$P_k(s,\, g')(x) = P_k(s,\, g+h)(x) = P_k(s,\, g)(x).$$

REMARK 1. We observe that induction on (4) yields that if $s\in\Gamma_c^\infty(E)$, g, $h\in C^\infty(M:R)$, $d(g+th)\neq 0$ on supp s for $0\leqq t\leqq 1$ and if $x\in M$ then, for each integer $N>0$,

$$P_k(s, g+h)(x) = \sum_{l<N} \frac{i^l}{l!} P_{k+(l)}((h-h(x))^l s, g)(x)$$
$$+ i^N \int_0^1 \cdots \int_0^1 t_1^{N-1} t_2^{N-2} \cdots t_{N-1}$$
$$\cdot P_{k+(N)}((h-h(x))^N s, g+t_1t_2 \cdots t_N h)(x) dt_1 \cdots dt_N$$

where $k+(l)=k++ \cdots +$ (l times). Suppose now that h vanishes of order 2 at x. If we choose $N>n_k$ then the integral term vanishes and it follows that $P_k(s, g)$ is polynomial in the derivatives of g of order $\geqq 2$.

REMARK 2. In [1] Hörmander shows that pseudo-differential operators admit formal transposes. Suppose that $P:\Gamma_c^\infty(E)\to\Gamma^\infty(F)$ is a pseudo-differential operator and suppose that dv is a smooth density on M. Then the *formal transpose* of P is the unique pseudo-differential operator $P':\Gamma_c^\infty(F^*)\to\Gamma^\infty(E^*)$ such that

$$\int_M \langle P'\omega, s\rangle dv = \int_M \langle \omega, Ps\rangle dv \tag{6}$$

for $s\in\Gamma_c^\infty(E)$, $\omega\in\Gamma_c^\infty(F^*)$. From (6) and the uniqueness of the asymptotic expansions of P and P' we see that the expansions have the same exponents r_k and

$$\int_M \langle P_k'(\omega, g), s\rangle dv = \int_M \langle \omega, P_k(s, -g)\rangle dv$$

for $s\in\Gamma_c^\infty(E)$, $\omega\in\Gamma_c^\infty(F^*)$, $g\in C^\infty(M:R)$, $dg\neq 0$ on supp $s\cup$supp ω. Thus the differential operator $\omega\to P_k'(\omega, g)$ is just the formal transpose of the differential operator $s\to P_k(s, -g)$.

REFERENCES

1. L. Hörmander, *Pseudo-differential operators*, Comm. Pure Appl. Math. **18** (1965), 501–517. MR **31** #4970.

2. R. S. Palais et al., *Seminar on the Atiyah-Singer index theorem*, Ann. of Math. Studies, no. 57, Princeton Univ. Press, Princeton, N. J., 1965. MR **33** #6649.

3. B. E. Petersen, *On the calculus of symbols for pseudo-differential operators*, Proc. Sympos. Pure Math., vol. 16, Amer. Math. Soc., Providence, R. I., 1970, pp. 169–173.

4. W. Shih, *On the symbol of a pseudo-differential operator*, Bull. Amer. Math. Soc. **74** (1968), 657–659. MR **37** #2249.

OREGON STATE UNIVERSITY, CORVALLIS, OREGON 97331