THE TOTAL SYMBOL OF A PSEUDO-DIFFERENTIAL OPERATOR

BENT E. PETERSEN

ABSTRACT. This note presents a direct completely coordinate-free proof that the symbols of a pseudo-differential operator are differential operators whose coefficients are functions on the higher order cotangent bundles.

Let $M$ be a smooth manifold and let $E$ and $F$ be complex smooth vector bundles on $M$ (smooth = $C^\infty$). In [1] L. Hörmander defines a pseudo-differential operator from $E$ to $F$ to be a continuous linear map $P: \Gamma_x^\infty(E) \to \Gamma_x^\infty(F)$ such that there exists a sequence $m_0 > m_1 > \cdots \to -\infty$ of real numbers such that we have an asymptotic expansion

\begin{equation}
  e^{-\lambda \varphi} P(e^{\lambda \varphi} s) \sim \sum_{k=0}^\infty P_k(s, g) \lambda^{m_k}
\end{equation}

in the following sense. If $s \in \Gamma_x^\infty(E)$ and if $K$ is a compact subset of $C^\infty(M; \mathbb{R})$ such that $g \in K$ implies $dg \neq 0$ on supp $s$ then, for each integer $N > 0$,

\begin{equation}
  \left\{ \lambda^{-m_N} \left( e^{-\lambda \varphi} P(e^{\lambda \varphi} s) - \sum_{k < N} P_k(s, g) \lambda^{m_k} \right) : \lambda \geq 1, g \in K \right\}
\end{equation}

is a bounded subset of $\Gamma_x^\infty(F)$. Here $\Gamma_x^\infty(F)$ denotes the smooth sections of $F$, $\Gamma_x^\infty(E)$ the compactly supported smooth sections of $E$, and $C^\infty(M; \mathbb{R})$ the smooth real-valued functions on $M$. All these spaces are provided with the usual Schwartz topologies.

A few properties of the asymptotic expansion are immediate. From (2) we see that, for each integer $n \geq 0$,

\begin{equation}
  P_n(s, g) = \lim_{\lambda \to \infty} \lambda^{-n} \left( e^{-\lambda \varphi} P(e^{\lambda \varphi} s) - \sum_{k < n} P_k(s, g) \lambda^{m_k} \right)
\end{equation}

where the convergence is in the topology of $\Gamma_x^\infty(F)$, uniformly for $g \in K$. By induction it follows that $P_n(s, g)$ depends continuously on $g$ for $g \in K$. From (3) we also have that the expansion (1) is unique and that $P_n(s, g)$ is positively homogeneous in $g$ of degree $m_n$, and linear in $s$ for supp $s \subseteq \{ x \in M : dg(x) \neq 0 \}$. Now suppose $A$ is a com-
pact subset of $M$ and let $K$ be a compact subset of $\mathcal{C}^\omega(M;\mathbb{R})$ such that $g \in K$ implies $dg \not\equiv 0$ on $A$. The subspace $\Gamma^\omega(E | A)$ of $\Gamma^\omega_*(E)$ consisting of sections with supports in $A$ is a Fréchet space. Then by the Banach-Steinhaus theorem and induction on (3), $s \mapsto P_n(s, g)$ is a continuous linear map of $\Gamma^\omega(E | A)$ into $\Gamma^\omega(F)$ for each $g \in K$ and we have the following stronger version of (2). If $B$ is a bounded subset of $\Gamma^\omega_*(E)$ and $K$ is a compact subset of $\mathcal{C}^\omega(M;\mathbb{R})$ and $c > 0$, such that if $g \in K$ and $s \in B$ then $|dg| \leq c$ on supp $s$, then, for each integer $N > 0$,

$$
(2)' \quad \left\{ \lambda^{-r_N} \left( e^{-\beta s} P(e^{\beta s}) - \sum_{k < N} P_k(s, g) \lambda^r_k \right) : \lambda \geq 1, \ g \in K, \ s \in B \right\}
$$

is a bounded subset of $\Gamma^\omega(F)$.

The purpose of this note is to give a completely coordinate-free proof of the following theorem.

**Theorem.** Let $P$ be a pseudo-differential operator with asymptotic expansion (1). For each integer $k \geq 0$ there exists a unique fibre-preserving smooth map

$$
\sigma_k(P): T^*_{k+1} \rightarrow \text{Hom}(J^k(E), F)
$$

positively homogeneous of degree $r_k$, where $n_k$ is the greatest integer in $r_0 - r_k$, such that if $s \in \Gamma^\omega_*(E)$, $g \in \mathcal{C}^\omega(M;\mathbb{R})$, $dg \not\equiv 0$ on supp $s$, then

$$
\sigma_k(P) \cdot j_n(g) \cdot j_n(s) = P_k(s, g).
$$

In particular

$$
\sigma_0(P): T^* \rightarrow \text{Hom}(E, F) \quad \text{and} \quad \sigma_0(P) \cdot dg \cdot s = P_0(s, g).
$$

This theorem has previously been announced in [3] and [4] without proof. $\sigma_k(P)$ is called the $k$th symbol of $P$ and $\sigma_0(P)$ is called the top order symbol or frequently just the symbol of $P$. A proof of the theorem may be based on the explicit coordinate expressions given in [1] for the $P_k$. These expressions however follow from some delicate Fourier transform estimates whereas the present method involves only some simple manipulations of asymptotic series.

We begin by explaining some of the notation. $J^n(E)$ denotes the $n$th jet bundle of $E$ [2, Chapter 4], and $j_n$ is the $n$th jet extension map. $T^n$ is the $n$th order cotangent bundle of $M$ and may conveniently be defined as the kernel of the canonical morphism $J^n(1) \rightarrow 1$ where 1 denotes the trivial real line bundle. It is easy to see that $T^n$ may be regarded as a subbundle of $J^{n-1}(T^*)$ and that if $g \in \mathcal{C}^\omega(M;\mathbb{R})$ then $j_{n-1}(dg)$ is a section of $T^n$ and that such sections generate the fibres. For each $n \geq 1$ we have a natural morphism $\pi_n: T^n \rightarrow T^*$ and we now
define $T_0^* = \pi_n^{-1}(T^* - (0))$. In particular $T_0^* = T^* - (0)$. Here $T^* - (0)$ denotes the complement of the zero section in $T^*$.

The theorem then says that $P_k(s, g)$ is a linear differential operator of degree $\leq n_k$ in $s$ with coefficients that are (nonlinear) differential operators in $dg$ of degree $\leq n_k$. At the end of the proof we will show that whereas the dependence on $dg$ may be highly nonlinear, the derivatives of $dg$ enter polynomially. The following lemma is the key to the proof. We note that by adding some zero terms to the expansion (1) we may assume that for each integer $k \geq 0$ there exists an integer $k + > k$ such that $r_{k+} = r_k - 1$.

**Lemma.** Let $s \in \Gamma^w(E), g, h \in C^\infty(M; R)$ and suppose that $d(g + th) \neq 0$ on supp $s$ for $0 \leq t \leq 1$. Then for each $x \in M$ we have

\[
(4) \quad P_k(s, g + h)(x) - P_k(s, g)(x) = \int_0^1 P_{k+}((h - h(x))s, g + th)(x)dt.
\]

To prove the lemma note that

\[
\lambda^{-r_N} \left( e^{-\lambda h(t+k)} P(e^{\lambda h}(g+h)(h - h(x))s) - \sum_{k < N} P_k((h - h(x))s, g + th)\lambda^{r_k} \right)
\]

when evaluated at $x$ is a bounded set in $C([0, 1], \mathbb{F}_x)$ for $\lambda \geq 1$. If we integrate over $t$ we obtain a bounded set in $F_x$, i.e. we have an asymptotic expansion

\[
eq e^{-\lambda h(x)} \int_0^1 P(e^{\lambda h}(h - h(x))s, g + th)(x)dt
\]

But now

\[
eq e^{-\lambda h(x)}(e^{\lambda h(x)} - 1)s = i\lambda \int_0^1 e^{\lambda h}(h - h(x))e^{\lambda h(x)}dt
\]

where the integral converges in $\Gamma^w(E)$, since the integrand as a function of $t$ belongs to $C([0, 1], \Gamma^w(E))$. Thus $P$ commutes with the integral and therefore

\[
eq e^{-\lambda h(x)} P(e^{\lambda h}(h - h(x))s(x) - e^{\lambda h}(g + h)(x)) - 1)s(x)
\]

\[
= e^{-\lambda h(x)} P(e^{\lambda h}(h - h(x))s(x) - 1)s(x)
\]

\[
\sim \sum_{k=0}^{\infty} \lambda^{r_k+1} \int_0^1 P_k((h - h(x))s, g + th)(x)dt
\]
which completes the proof of the Lemma. We now prove the Theorem.

Suppose \( s \in \Gamma_\epsilon^\omega(E), \ g \in C^\omega(M : \mathbb{R}) \) and \( x \in M \). Suppose \( h \in C^\omega(M : \mathbb{R}), \ h(x) = 0 \) and \( d(g + th) \neq 0 \) on \( \text{supp} \ s \) for \( 0 \leq t \leq 1 \). Replacing \( h \) by \( rh \) and \( t \) by \( tr^{-1} \) \( (0 < r < 1) \) in (4) by continuity of the argument of the integral we see that

\[
iP_{k+}(hs, g)(x) = \frac{d}{dt}P_k(s, g + th)(x) \bigg|_{t=0}.
\]

Putting it differently, if for each integer \( k \geq 0 \) we define \( P_{k-}(s, g) \) by

\[
P_{k-}(s, g) = \begin{cases} P_l(s, g) & \text{if } l = k, \\ 0 & \text{otherwise}, \end{cases}
\]

then

\[
iP_k(hs, g)(x) = \frac{d}{dt}P_{k-}(s, g + th)(x) \bigg|_{t=0}
\]

where (5) may be seen to hold for those \( k \) not of the form \( l+ \) merely by adding sufficient zero terms to the expansion (1) before going through the above arguments.

Suppose now that we have shown that \( P_{k-} \) is a differential operator in \( s \) of degree \( \leq N \). If \( s \) vanishes at \( x \) of order \( N+1 \) then \( P_{k-}(s, g + th)(x) = 0 \) and hence, by (5), \( P_k(hs, g)(x) = 0 \). Thus \( P_k \) is a differential operator in \( s \) of degree \( \leq N+1 \). Since \( P_{k-} = 0 \) if \( r_0 - r_k < 1 \) it follows by induction that \( P_k \) is a differential operator in \( s \) of degree \( \leq n_k \). In particular \( P_k \) is local in \( s \).

Now suppose that \( s \in \Gamma_\epsilon^\omega(E), \ g, \ g' \in C^\omega(M : \mathbb{R}) \) with \( dg \neq 0 \) and \( dg' \neq 0 \) on \( \text{supp} \ s \) and suppose that for some \( x \in M \) we have \( j_n(s)(dg')(x) = n_k(dg')(x) \). We wish to show that \( P_k(s, g)(x) = P_k(s, g')(x) \). If \( x \notin \text{supp} \ s \) we are done since \( P_k \) is local in \( s \). If \( x \in \text{supp} \ s \) since \( dg(x) = dg'(x) \) and \( P_k \) is local in \( s \) we may cut down the support of \( s \) so that \( (1-t)dg + tdg' \neq 0 \) on \( \text{supp} \ s \) for \( 0 \leq t \leq 1 \). Now let \( h = g - g' \) so \( h - h(x) \) vanishes of order at least \( n_k + 2 = n_k + 1 \) at \( x \). Then

\[
P_{k+}(h - h(x)s, g + th)(x) = 0
\]

for \( 0 \leq t \leq 1 \), since \( P_{k+} \) is a differential operator in \( s \) of degree \( \leq n_{k+} \). Thus by (4) we have

\[
P_k(s, g')(x) = P_k(s, g + h)(x) = P_k(s, g)(x).
\]

Remark 1. We observe that induction on (4) yields that if \( s \in \Gamma_\epsilon^\omega(E), \ g, \ h \in C^\omega(M : \mathbb{R}), \ d(g + th) \neq 0 \) on \( \text{supp} \ s \) for \( 0 \leq t \leq 1 \) and if \( x \in M \) then, for each integer \( N > 0 \),
\[ P_k(s, g + h)(x) = \sum_{i < N} \frac{i!}{l!} P_{k+l(i)}((h - h(x))^l s, g)(x) \]

\[ + \int_0^1 \cdots \int_0^1 t_{N-2} \cdots t_{N-1} \]

\[ \cdot P_{k+(N)}((h - h(x))^N s, g + t_1 t_2 \cdots t_N h)(x) dt_1 \cdots dt_N \]

where \( k + (l) = k + \cdots + (l \text{ times}) \). Suppose now that \( h \) vanishes of order 2 at \( x \). If we choose \( N > n_k \) then the integral term vanishes and it follows that \( P_k(s, g) \) is polynomial in the derivatives of \( g \) of order \( \geq 2 \).

**Remark 2.** In [1] Hörmander shows that pseudo-differential operators admit formal transposes. Suppose that \( P : \Gamma^s_c(E) \rightarrow \Gamma^s(F) \) is a pseudo-differential operator and suppose that \( dv \) is a smooth density on \( M \). Then the **formal transpose** of \( P \) is the unique pseudo-differential operator \( P' : \Gamma^s_c(F^*) \rightarrow \Gamma^s(E^*) \) such that

\[ \int_M \langle P' \omega, s \rangle dv = \int_M \langle \omega, Ps \rangle dv \]

for \( s \in \Gamma^s_c(E), \omega \in \Gamma^s_c(F^*) \). From (6) and the uniqueness of the asymptotic expansions of \( P \) and \( P' \) we see that the expansions have the same exponents \( r_k \) and

\[ \int_M \langle P'_k(\omega, g), s \rangle dv = \int_M \langle \omega, P_k(s, \omega) \rangle dv \]

for \( s \in \Gamma^s_c(E), \omega \in \Gamma^s_c(F^*), g \in C^\infty(M; R), dg \neq 0 \) on \( \text{supp } s \cup \text{supp } \omega \). Thus the differential operator \( \omega \rightarrow P'_k(\omega, g) \) is just the formal transpose of the differential operator \( s \rightarrow P_k(s, \omega) \).

**References**


Oregon State University, Corvallis, Oregon 97331