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SEMICONTINUITY OF NULLITY OR DEFICIENCY IMPLIES NORMABILITY OF THE SPACE

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ABSTRACT. In this paper the upper semicontinuity of nullity and deficiency on locally convex spaces is examined. If either is semicontinuous in the topology of uniform convergence on bounded sets on $L(X)$, then X is normable. If the invertible elements in $L(X)$ are open, then X is normable. The results are applied to topological algebras.

Let X be a locally convex Hausdorff linear topological space. Let $L(X)$ be the space of all continuous linear maps from X into X with the topology of uniform convergence on bounded sets. If $T\in L(X)$, define $\tilde{T}:X/\ker T\to X$ by $\tilde{T}(x+\ker T)=Tx$. Let $X/\ker T$ have the quotient topology. There is a $\tilde{T}^{-1}:R(T)\to X/\ker T$. $\tilde{T}^{-1}$ is continuous if ond only if (for a normed space) $\gamma(T)=\inf\{|Tx|/d(x, \ker T)\}$, the (reduced) minimum modulus of T, satisfies $\gamma(T)>0$, if and only if (for a Banach space) $R(T)$ is closed.

If X is normable, then several results on the semicontinuity of the nullity $\alpha(T)=\dim \ker T$ and the deficiency $\beta(T)=\operatorname{codim} \operatorname{cl}(R(T))$ are known, such as the following result due to Webb.

PROPOSITION [6]. *Let X be a normed linear space and $T\in L(X)$ with $\tilde{T}^{-1}$ continuous. Then there is a neighborhood N of T in $L(X)$ such that, for any $A\in N$,*

(1) $\alpha(A)\leqq\alpha(T)$,

(2) $\beta(A)\leqq\beta(T)$.

COROLLARY. *Let X be a normed linear space. There is a neighborhood N of $I\in L(X)$ such that, for any $A\in N$,*

(1) $\alpha(A)=0$,

(2) $\beta(A)=0$.

We shall show that if these conclusions hold then the space X must be normable.

PROPOSITION. *Let X be a l.c.s. If there is a neighborhood N of $I\in L(X)$ for the topology of uniform convergence on bounded sets such that for $A\in N$ either $\alpha(A)=0$ or $\beta(A)=0$, then X is normable.*

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PROOF. There is a closed, convex, circled and bounded set B and a convex circled neighborhood of zero V such that if $A\in N(B,V) =\{A\in L(X):AB\subseteq V\}$ then either $\alpha(I-A)=0$ or $\beta(I-A)=0$.

Assume if possible that there is an element $x_0\in V\backslash B$. Then $x_0\neq 0$. By the separation theorems [2] there is a continuous linear functional m' on X such that $|m'(b)|\leqq|m'(x_0)|>0$ for all $b\in B$. Set $m(x) =m'(x)/m'(x_0)$. Then $|m(b)|\leqq 1$ for all $b\in B$ and $m(x_0)=1$. Define $Ax=m(x)x_0$. $A\in L(X)$ and since $|m(b)|\leqq 1$ and V is circled, $Ab =m(b)x_0\in V$ for all $b\in B$. Thus $A\in N(B,V)$. $(I-A)x_0=0$, so $\alpha(I-A)\neq 0$. If $(I-A)x=x_0$, then $x-m(x)x_0=x_0$, $x=(1+m(x))x_0$, and $(I-A)x=0$. Thus $x_0\notin R(I-A)$. Since $R(I-A)$ is closed [3], $\beta(I-A)\neq 0$. Thus we must have $V\subseteq B$, so V is a bounded neighborhood of zero, and X is normable. □

COROLLARY. *X is normable if and only if for every $T\in L(X)$ with $\tilde{T}^{-1}$ continuous there is a neighborhood N of T in $L(X)$ such that for $A\in N$ either*

(1) $\alpha(A)\leqq\alpha(T)$ *or*

(2) $\beta(A)\leqq\beta(T)$.

COROLLARY. *X is normable if and only if the invertible elements in $L(X)$ are open in the topology of uniform convergence on bounded sets.*

It should be noted that perturbations of I may be invertible for operators in some subset of $L(X)$. For example, the following is a corollary of a result of Vladimirskii [5].

PROPOSITION. *Let X be a complete l.c.s. Let U be a closed absolutely convex neighborhood of zero. Then there is a closed absolutely convex neighborhood of zero V such that if $\alpha\in L(X)$ is open, $\alpha(U)$ is bounded, and $\alpha(U)\subseteq V$ then $I+\alpha$ is invertible.*

Results related to the above corollary have been obtained by Blair [1] and Williamson [7]. X is normable if and only if there is a subalgebra A of $L(X)$ containing all operators of finite rank such that multiplication $A\times A\to A$ is continuous for the topology of uniform convergence on bounded sets, compact sets, or pointwise convergence on A. X is normable if and only if there is a linear set L in $L(X)$ containing all operators of finite rank which can be given a linear topology such that $(T,x)\to Tx$ is continuous.

The following application to topological algebras (locally convex topological rings with identity) was suggested by this paper's referee. A topological algebra X has a continuous inverse [4] if there is a neighborhood W of the identity e such that every element of W has an inverse and the mapping $x \to x^{-1}$ is continuous on W. In this case the invertible elements are open and $x \to x^{-1}$ is continuous. The left regular representation of X is $R: X \to L(X)$ defined by $R(a)x = ax$.

PROPOSITION. *Let X be a topological algebra whose left regular representation contains the operators of finite rank. If X has a continuous inverse, then X is normable.*

PROOF. The map $a \to R(a)$ gives $R(X)$ the topology of pointwise convergence. There is a neighborhood U of $I = R(e)$ for the stronger topology of uniform convergence on bounded sets such that $U \cap R(X) \subseteq R(W)$, whose elements are invertible. Thus there is a closed, convex, circled and bounded set $B \subseteq X$ and a convex circled neighborhood V of 0 such that $I + N(B, V) \cap R(X) \subseteq R(W)$. If there is an $x_0 \in V \setminus B$, then as before there is a continuous linear functional m such that $|m(B)| \leqq 1$ and $m(x_0) = 1$. If $Ax = m(x)x_0$, then $-A \in N(B, V) \cap R(X)$ and $(I - A)x_0 = 0$. Thus $I - A \in R(W)$ is not invertible, and so $V \subseteq B$ and X is normable. □

Applying the left regular representation to the results of Blair and Williamson [7], we obtain the following result.

PROPOSITION. *Let X be a topological algebra whose left regular representation contains the operators of finite rank. Then X is normable if and only if multiplication $X \times X \to X$ is continuous.*

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