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IRREDUCIBLE ALGEBRAS OF OPERATORS WHICH CONTAIN A MINIMAL IDEMPOTENT

BRUCE A. BARNES[1]

ABSTRACT. We prove that when A is a closed subalgebra of the bounded operators on a reflexive Banach space X, which acts irreducibly on X and contains a minimal idempotent, then every bounded operator with finite dimensional range on X is in A. We use this result to prove that every continuous irreducible representation of a GCR-algebra on a Hilbert space $\mathcal{H}$ is similar to a *-representation on $\mathcal{H}$.

1. **Notation and terminology.** Assume that X is a normed linear space. $\mathcal{B}(X)$ denotes the algebra of bounded operators on X, $\mathcal{C}(X)$ denotes the algebra of compact operators on X, and $\mathcal{F}(X)$ denotes the algebra of bounded operators on X which have finite dimensional range. A subalgebra A of $\mathcal{B}(X)$ acts irreducibly on X (or is irreducible on X) if the only closed A-invariant subspaces of X are 0 and X. A acts strictly irreducibly on X if the only A-invariant subspaces of X are 0 and X.

If A is a normed algebra and X is a normed linear space, then a representation of A on X is an algebra homomorphism π of A into $\mathcal{B}(X)$. A representation π of A on X is irreducible (strictly irreducible) if $\pi(A)$ acts irreducibly (strictly irreducibly) on X. When A has an involution $*$ and $\mathcal{H}$ is a Hilbert space, a representation π of A on $\mathcal{H}$ is a *-representation if $\pi(a^*)=\pi(a)^*$ for all $a\in A$.

Let X be a normed linear space. We denote the dual space of X as X^*. Given $x\in X$ and $f\in X^*$, let $(f|x)$ be the operator defined by $(f|x)(y)=f(y)\cdot x$, $y\in X$. Every bounded operator on X with 1-dimensional range has the form $(f|x)$ for some $x\in X, f\in X^*$. When $\mathcal{H}$ is a Hilbert space and $\phi, \psi\in\mathcal{H}$, then $(\phi|\psi)$ is the operator defined by $(\phi|\psi)(\tau)=(\tau,\phi)\cdot\psi$, $\tau\in\mathcal{H}$.

A nonzero idempotent E in a complex normed algebra A is a minimal idempotent of A if $EAE=\{\lambda E|\lambda \text{ complex}\}$. There is a close relationship between minimal idempotents of A and minimal left (or right) ideals of A; see [**5**, pp. 45–46]. All vector spaces in this paper are complex.

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2. **Irreducible algebras of operators which contain a minimal idempotent.** We assume throughout this section that X is a normed linear space and that A is a subalgebra of $\mathcal{B}(X)$ such that A acts irreducibly on X. We prove several lemmas and then the main result of this section (Theorem 3).

LEMMA 1. *If E is a minimal idempotent of A, then there exist $x\in X$, $f\in X^*$, with $f(x)=1$, such that $E=(f|x)$.*

PROOF. Choose $x\in X$, $x\neq 0$, such that $E(x)=x$. Let $Y=\{T(x)\mid T\in A\}$. Y is an invariant subspace of X for A. When $T\in A$ we denote by T' the restriction of T to Y. Assume that $T_1(x)$ and $T_2(x)$ are in the range of E', T_1, $T_2\in A$. Then $T_i(x)=E'T_i(x)$ $=ET_iE(x)=\lambda_i x$ for some scalars λ_i, $i=1$, 2. Thus E' has 1-dimensional range on Y. It follows that $E'=(g|x)$ for some $g\in Y^*$. Also since E' is a projection, then $g(x)=1$. Let f be the unique extension of g to a continuous functional on X (note that Y is dense in X). Then $E=(f|x)$ since $E-(f|x)$ is 0 on Y.

LEMMA 2. *Assume that A is a closed subalgebra of $\mathcal{B}(X)$ and that A contains a minimal idempotent E. Then A acts strictly irreducibly on X.*

PROOF. By Lemma 1, $E=(f|x)$ for some $x\in X$, $f\in X^*$, with $f(x)$ $=1$. $AE=\{(f|T(x))\mid T\in A\}$ is a left ideal of A which is closed in the operator norm. $W=\{T(x)\mid T\in A\}$ is a nonzero invariant subspace of X for A, and, therefore, W is dense in X. Given $y\in X$ we can choose $\{y_n\}\subset W$ such that $y_n=T_n(x)\to y$. Then $(f|y_n)=T_n(f|x)$ $\in AE$ for all n and $(f|y_n)\to(f|y)$ in the operator norm. Therefore $(f|y)\in AE$. Assume now that y_1, $y_2\in X$ and $y_1\neq 0$. As above $(f|y_1)$, $(f|y_2)\in AE$. $\mathcal{L}=A(f|y_1)$ is a nonzero left ideal of A in AE. By the proof of [**5**, Lemma (2.18), p. 45], $\mathcal{L}=AE$. Then there exists $T\in A$ such that $T(f|y_1)=(f|y_2)$. It follows that $T(y_1)=y_2$, and therefore A acts strictly irreducibly on X.

THEOREM 3. *Assume that X is a reflexive Banach space, and that A is a closed subalgebra of $\mathcal{B}(X)$ which acts irreducibly on X and contains a minimal idempotent E. Then $\mathfrak{F}(X)\subset A$.*

PROOF. By Lemma 1 there exists $x\in X$, $f\in X^*$ with $f(x)=1$ such that $E=(f|x)$. By Lemma 2, A acts strictly irreducibly on X. Therefore $(f|y)\in A$ for all $y\in X$. Let $K=\{g\in X^*\mid (g|y)\in A$ for some $y\in X$, $y\neq 0\}$. We prove that K is a closed subspace of X^*. For assume that f_1, $f_2\in K$. Then there exists $x_1\neq 0$, $x_2\neq 0$ in X such that $(f_1|x_1)$ and $(f_2|x_2)$ are in A. Choose $T\in A$ such that $T(x_1)=x_2$. Then $(f_1+f_2|x_2)=T(f_1|x_1)+(f_2|x_2)$. Thus, $f_1+f_2\in K$. Now assume that

$(f_n|x_n)\in A$, $x_n\neq 0$ for all n, and $f_n\to g$ in X^*. Choose $T_n\in A$ such that $T_n(x_n)=x$ for all n. Then $T_n(f_n|x_n)=(f_n|x)\in A$ for all n and $(f_n|x)\to(g|x)$ in $\mathfrak{B}(X)$. Since A is closed, $(g|x)\in A$, and this proves $g\in K$.

Now suppose that $K\neq X^*$. Since K is closed and X is reflexive, there exists $y\in X$, $y\neq 0$, such that $g(y)\neq 0$ for all $g\in K$. Given $T\in\mathfrak{B}(X)$, let T^* be the conjugate (adjoint) operator of T on X^*. Note that $(f|x)T=(T^*(f)|x)$. Then $T^*(f)\in K$ for all $T\in A$, and it follows that $T^*(f)(y)=f(T(y))=0$ for all $T\in A$. But there exists $T\in A$ such that $T(y)=x$, and $f(x)=1$. This contradiction proves that $K=X^*$. Therefore, by the definition of K, A contains every bounded operator on X which has 1-dimensional range. This implies that $\mathfrak{F}(X)\subset A$, so the proof of the theorem is complete.

We denote the radical of an algebra B by $\mathrm{rad}(B)$. In the next theorem we give a sufficient condition that an irreducible algebra A contain a minimal idempotent.

THEOREM 4. *Assume that X is a Banach space and that A is a closed subalgebra of $\mathfrak{B}(X)$ which acts irreducibly on X. If A contains an operator $C\in\mathfrak{C}(X)$ such that C does not have zero spectrum, then A contains a minimal idempotent.*

PROOF. $A\cap\mathfrak{C}(X)$ is a closed subalgebra of $\mathfrak{C}(X)$ which contains C. Since C does not have zero spectrum, we can produce a nonzero projection $F\in A\cap\mathfrak{C}(X)$ by taking the appropriate contour integral about a nonzero (isolated) point of the spectrum of C. F must have finite dimensional range. Then FAF is a finite dimensional subalgebra of A. By the Wedderburn theory there exists a projection E in FAF such that the residue class of E in the quotient algebra $FAF/\mathrm{rad}(FAF)$ is a minimal idempotent. $\mathrm{rad}(FAF)$ is nilpotent, so we can choose a positive integer m such that $(FSF)^m=0$ whenever $FSF\in\mathrm{rad}(FAF)$. By [2, Proposition 1, p. 48], $\mathrm{rad}(FAF)=F\,\mathrm{rad}(A)F$. Then if $T\in\mathrm{rad}(A)$, we have $ETE\in\mathrm{rad}(FAF)$. Therefore $(ETE)^m=0$ whenever $T\in\mathrm{rad}(A)$.

Now we show that $\mathrm{rad}(A)\cap AE=0$. For suppose not. Then there exists $T\in\mathrm{rad}(A)\cap AE$ such that $T=TE\neq 0$. Choose $x\in X$, $x\neq 0$, such that $E(x)=x$ and $T(x)\neq 0$. Set $M=\mathrm{rad}(A)\cap AE$. If $S\in M$, then $S^{m+1}=S(ESE)^m=0$. The set $\{S(x)\,|\,S\in M\}$ is a nonzero invariant subspace of X for A. Therefore there exists $\{S_n\}\subset M$ such that $S_n(x)\to x$. $EM(x)$ is a finite dimensional (and hence closed) subspace of X. Since $ES_n(x)\to E(x)=x$, $x\in EM(x)$. Therefore there exists $S\in M$ such that $ES(x)=x$. But $ES\in M$, so that $(ES)^{m+1}=0$. This is a contradiction since $0\neq x=(ES)^{m+1}(x)=0$. Therefore $\mathrm{rad}(A)\cap AE=0$.

Given $T\in A$ there exists a scalar λ and $S\in\text{rad}(FAF)$ such that $ETE=\lambda E+S$ (recall that $E+\text{rad}(FAF)$ is a minimal idempotent in $FAF/\text{rad}(FAF)$). Then $ETE-\lambda E=S\in\text{rad}(A)\cap AE=0$. Therefore E is a minimal idempotent of A.

COROLLARY 5. *Assume that A satisfies the hypotheses of Theorem 4. Then A acts strictly irreducibly on X.*

COROLLARY 6. *Assume that A satisfies the hypotheses of Theorem 4 and that X is a reflexive Banach space. Then $\mathfrak{F}(X)\subset A$.*

3. **Representations similar to *-representations.** In this section we give a sufficient condition that a continuous irreducible representation of a B^*-algebra on a Hilbert space $\mathcal{H}$ be similar to a *-representation on $\mathcal{H}$. R. V. Kadison gives very general necessary and sufficient conditions in [**3**]. Kadison also discusses the important connections this subject has with the representation theory of topological groups.

Throughout this section we assume that A is a B^*-algebra. S. Cleveland has shown, [**1**, Lemma 5.3, p. 1104], that when π is a continuous algebra isomorphism of A into a Banach algebra B, then $\pi(A)$ is closed in B. This is easily extended to the case where π has a nonzero kernel I. For since π is continuous, I is a closed ideal of A, and then A/I is again a B^*-algebra. Define $\tilde{\pi}$ on A/I by $\tilde{\pi}(a+I)=\pi(a)$, $a\in A$. $\tilde{\pi}$ is a continuous algebra isomorphism of A/I into B. Therefore, the range of $\tilde{\pi}$ is closed in B by Cleveland's result. But $\pi(A)=\tilde{\pi}(A)$, so that $\pi(A)$ is closed in B.

Throughout this section $\mathcal{H}$ denotes a Hilbert space. Assume that π is a continuous irreducible representation of the B^*-algebra A into $\mathcal{B}(\mathcal{H})$, and that $A/\ker(\pi)$ contains a minimal idempotent. Then $\pi(A)$ is an irreducible closed subalgebra of $\mathcal{B}(\mathcal{H})$ which contains a minimal idempotent. An application of Theorem 3 proves the following result.

LEMMA 7. *Assume that π is a continuous irreducible representation of A on $\mathcal{H}$ such that $A/\ker(\pi)$ contains a minimal idempotent. Then π is strictly irreducible on $\mathcal{H}$ and $\mathcal{C}(\mathcal{H})\subset\pi(A)$.*

Every minimal left ideal of A has the form Ah where h is a selfadjoint minimal idempotent by [**5**, Lemma (4.10.1), p. 261]. Assume that h is a selfadjoint minimal idempotent of A. Then hAh is just the set of all scalar multiples of h. We define an inner product $\langle\cdot,\cdot\rangle$ on Ah by the rule $\langle xh, yh\rangle h=hy^*xh$; see [**5**, Theorem (4.10.3), p. 261]. We call $\langle\cdot,\cdot\rangle$ the canonical inner product on Ah. Now we prove the main result of this section.

THEOREM 8. *Assume that A is a B^*-algebra and that π is a continuous irreducible representation of A on $\mathcal{H}$. Assume that $A/\ker(\pi)$ contains a minimal left ideal. Then there exists a strictly irreducible *-representation ρ of A on $\mathcal{H}$ and a positive invertible operator $V \in \mathcal{B}(\mathcal{H})$ such that $\pi(a) = V^{-1}\rho(a)V$ for all $a \in A$.*

PROOF. We may assume without loss of generality that π is an isomorphism. For in the general case we can define $\tilde{\pi}$ on $A/\ker(\pi)$ as in the discussion preceding Lemma 7, and apply our arguments to $\tilde{\pi}$. Therefore we assume that π is an isomorphism and that A has a minimal left ideal Ah, where h is a selfadjoint minimal idempotent of A. Let $\langle \cdot, \cdot \rangle$ denote the canonical inner product on Ah and let $(\cdot, \cdot)$ denote the inner product on $\mathcal{H}$. $\pi(h)$ is a minimal idempotent of $\pi(A)$. It follows that there exists $\phi, \psi \in \mathcal{H}$ such that $(\psi, \phi) = 1$ and $\pi(h) = (\phi | \psi)$. By Lemma 7, π is strictly irreducible on $\mathcal{H}$. Since $\pi(Ah)\psi$ is a nonzero invariant subspace of $\mathcal{H}$ for $\pi(A)$, then $\mathcal{H} = \pi(Ah)\psi$. If, for some $x \in A$, $\pi(xh)\psi = 0$, then $\pi(x)(\phi|\psi)\psi = 0$, and therefore $\pi(x)\psi = 0$. Then $\pi(x)(\phi|\psi) = 0$, so that $\pi(xh) = 0$, and finally we have $xh = 0$. Therefore $xh \to \pi(xh)\psi$ is a 1-1 map of Ah onto $\mathcal{H}$. We use this map to transfer the canonical inner product on Ah to a positive definite form on $\mathcal{H}$. Define $[\pi(xh)\psi, \pi(yh)\psi] = \langle xh, yh \rangle$ for all $xh, yh \in Ah$. It follows from this definition that $[\cdot, \cdot]$ is a bounded positive definite form on $\mathcal{H}$. By Riesz's theorem there exists a positive operator $U \in \mathcal{B}(\mathcal{H})$ such that $[\phi_1, \phi_2] = (U\phi_1, \phi_2)$ for all $\phi_1, \phi_2 \in \mathcal{H}$. Assume that $\tau = \pi(xh)\psi \in \mathcal{H}$. Then

$$\begin{aligned}(\tau, \tau) &= (\pi(xh)\psi, \pi(xh)\psi) \leqq \|\pi(xh)\|^2\|\psi\|^2 \leqq \|\pi\|^2\|\psi\|^2\|xh\|^2 \\ &= \|\pi\|^2\|\psi\|^2\|hx^*xh\| = \|\pi\|^2\|\psi\|^2\|h\|\langle xh, xh\rangle.\end{aligned}$$

Set $M = \|\pi\|^2\|\psi\|^2\|h\|$. We have

$$(\tau, \tau) \leqq M\langle xh, xh\rangle = M[\pi(xh)\psi, \pi(xh)\psi] = M[\tau, \tau] = M(U\tau, \tau).$$

Therefore $U^{-1} \in \mathcal{B}(\mathcal{H})$.

Let $V = \sqrt{U}$. Define $\rho(a) = V\pi(a)V^{-1}$, $a \in A$. Given $\psi_1, \psi_2 \in \mathcal{H}$, there exists $\phi_1, \phi_2 \in \mathcal{H}$ and $x_1, x_2 \in A$ such that $\psi_i = V\phi_i$ and $\phi_i = \pi(x_ih)\psi$ for $i = 1, 2$. Then

$$\begin{aligned}(\rho(a)\psi_1, \psi_2) &= (V\pi(a)V^{-1}V\phi_1, V\phi_2) = [\pi(a)\phi_1, \phi_2] \\ &= [\pi(a)\pi(x_1h)\psi, \pi(x_2h)\psi] = \langle ax_1h, x_2h\rangle \\ &= \langle x_1h, a^*x_2h\rangle = [\pi(x_1h)\psi, \pi(a^*)\pi(x_2h)\psi] \\ &= (V\phi_1, V\pi(a^*)V^{-1}V\phi_2) = (\psi_1, \rho(a^*)\psi_2).\end{aligned}$$

Therefore ρ is a *-representation of A on $\mathcal{H}$. This completes the proof of the theorem.

Now assume that A is a GCR-algebra as defined by I. Kaplansky; see [**4**]. Assume that π is a continuous irreducible representation of A on a Hilbert space $\mathcal{H}$. Then $A/\ker(\pi)$ has no ideal divisors of zero by [**4**, Lemma 2.5, p. 223]. Therefore $A/\ker(\pi)$ contains a minimal left ideal by [**4**, Lemma 7.4, p. 247]. These remarks together with Theorem 8 prove the following corollary.

COROLLARY 9. *When A is a GCR-algebra, then every continuous irreducible representation of A on a Hilbert space $\mathcal{H}$ is similar to a strictly irreducible *-representation of A on $\mathcal{H}$.*

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UNIVERSITY OF OREGON, EUGENE, OREGON 97403